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Einladung zur Mathematik

Eine mathematische Einführung und Begleitung zum Studium der Physik und Informatik, Logos Verlag 2002

Lösungen zu den Übungsaufgaben

(Fehlerhinweise bitte an kurt.braeuer@uni-tuebingen.de)

Kapitel 5

Übungsbeispiel 5 (1)

a) $f(x) = 2x + 2 = 0 \quad \Rightarrow \quad 2x = -2 \quad \Rightarrow \quad x = -1$

b)

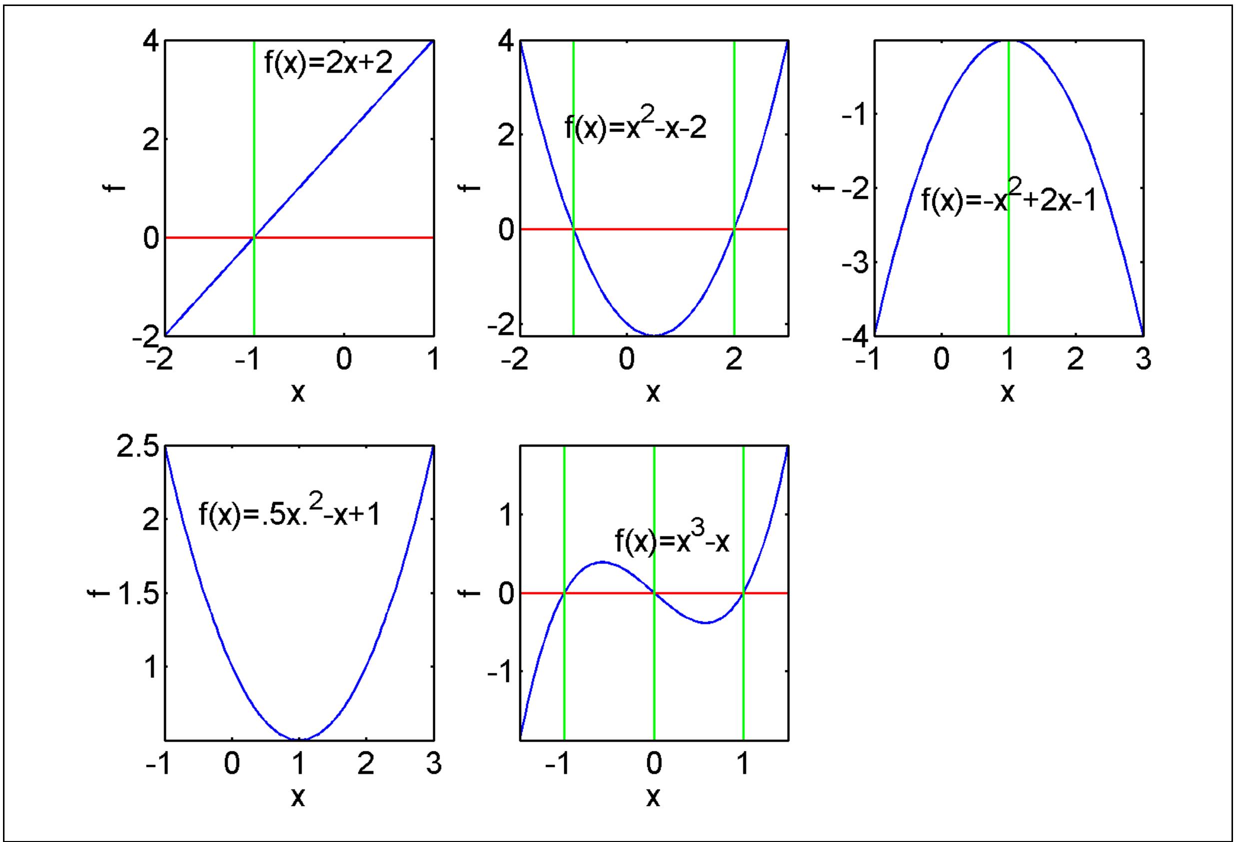
$$f(x) = x^2 - x - 2 = 0 \quad \Rightarrow \quad \underbrace{x^2 - 2x\tfrac{1}{2} + \tfrac{1}{4}}_{=\left(x-\frac{1}{2}\right)^2} - \tfrac{1}{4} - 2 = 0 \quad \Rightarrow \quad \left(x - \tfrac{1}{2}\right)^2 = \tfrac{9}{4}$$

$$\Rightarrow \quad x = \tfrac{1}{2} \mp \tfrac{3}{2} \quad \Rightarrow \quad \begin{cases} x_1 = -1 \\ x_2 = 2 \end{cases}$$

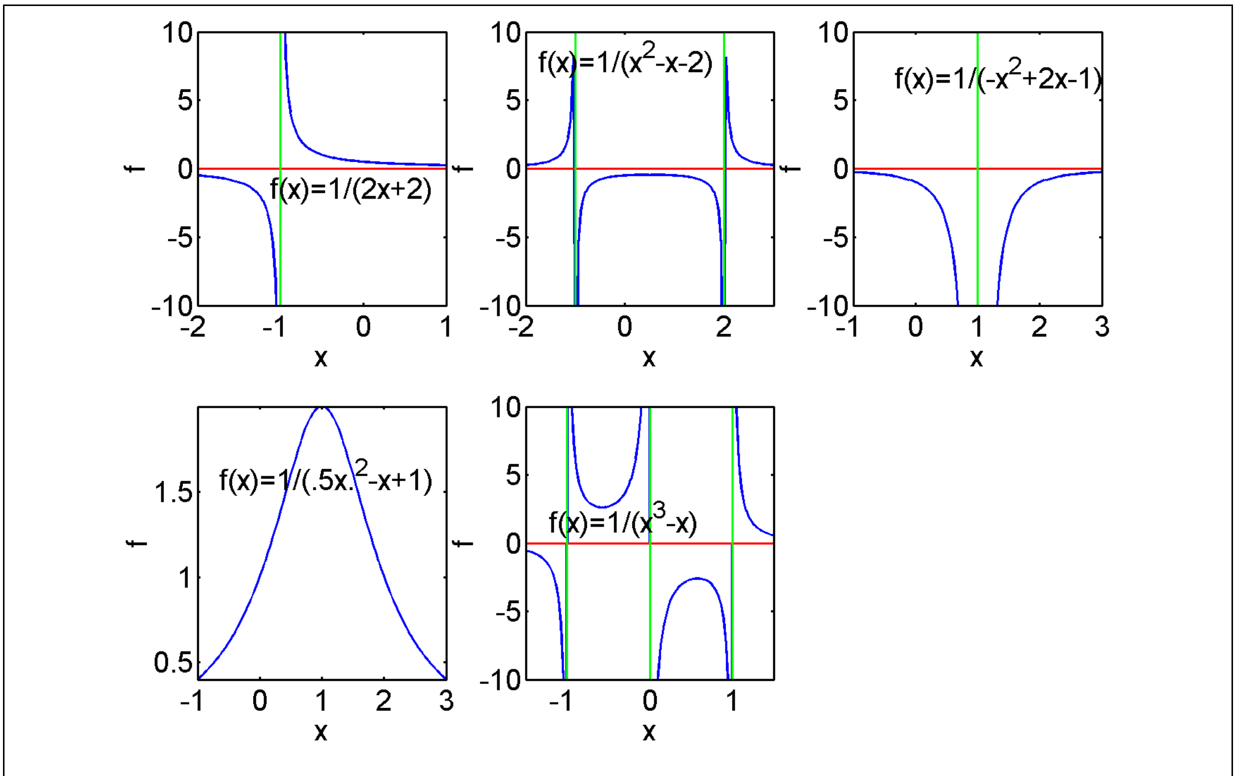
c) $f(x) = -x^2 + 2x - 1 = 0 \quad \Rightarrow \quad (x-1)^2 = 0 \quad \Rightarrow \quad x = 1$

d) $f(x) = \tfrac{1}{2}x^2 - x + 1 = 0 \quad \Rightarrow \quad \underbrace{x^2 - 2x + 1}_{=(x-1)^2} - 1 + 2 = 0 \quad \Rightarrow \quad (x-1)^2 = -1 \quad \Rightarrow \quad x = 1 \mp i$

e) $f(x) = x^3 - x = 0 \quad \Rightarrow \quad \begin{cases} x_0 = 0 \\ \underbrace{x^2 - 1}_{=(x+1)(x-1)} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} x_0 = 0 \\ x_+ = 1 \\ x_- = -1 \end{cases}$



Übungsbeispiel 5 (2)



Übungsbeispiel 5 (3)

$$\frac{3x^2-7x+1}{x-1}=ax+b+R$$

$$3x^2-7x+1=(x-1)(ax+b+R)=ax^2+(b-a)x-b+(x-1)R$$

$$\Rightarrow \begin{cases} a=3 \\ b-a=-7 \end{cases} \Rightarrow \begin{cases} a=3 \\ b=-4 \end{cases}$$

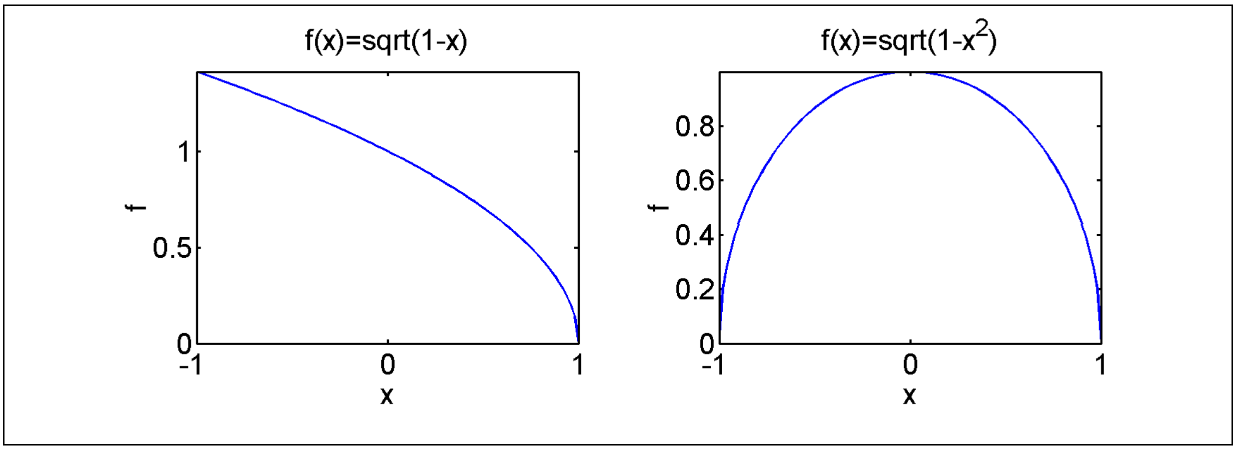
$$3x^2-7x+1=(x-1)(ax+b+R)=3x^2+-7x+4+(x-1)R$$

$$1=4+(x-1)R \quad \Rightarrow \quad R=-\frac{3}{(x-1)}$$

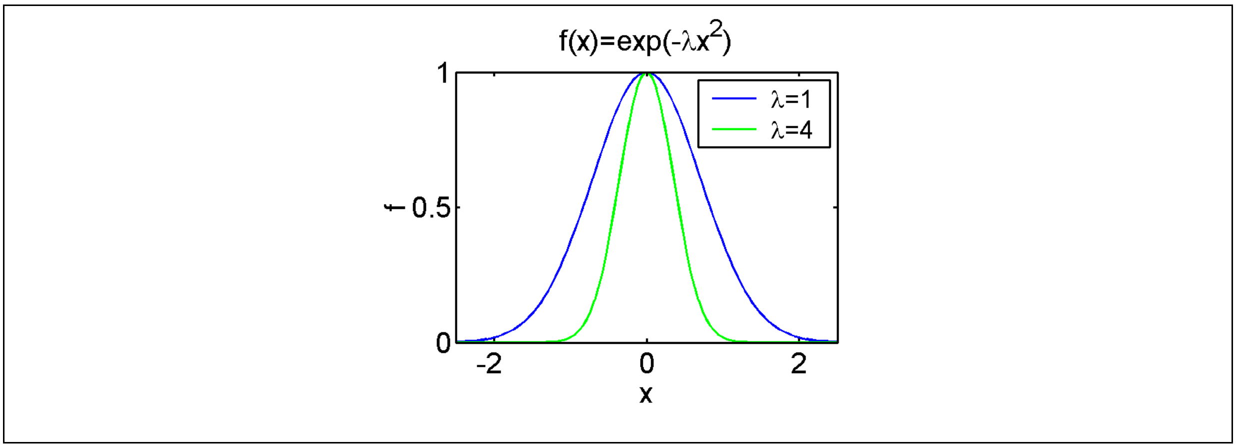
also

$$\frac{3x^2-7x+1}{x-1}=3x-4-\frac{3}{(x-1)}$$

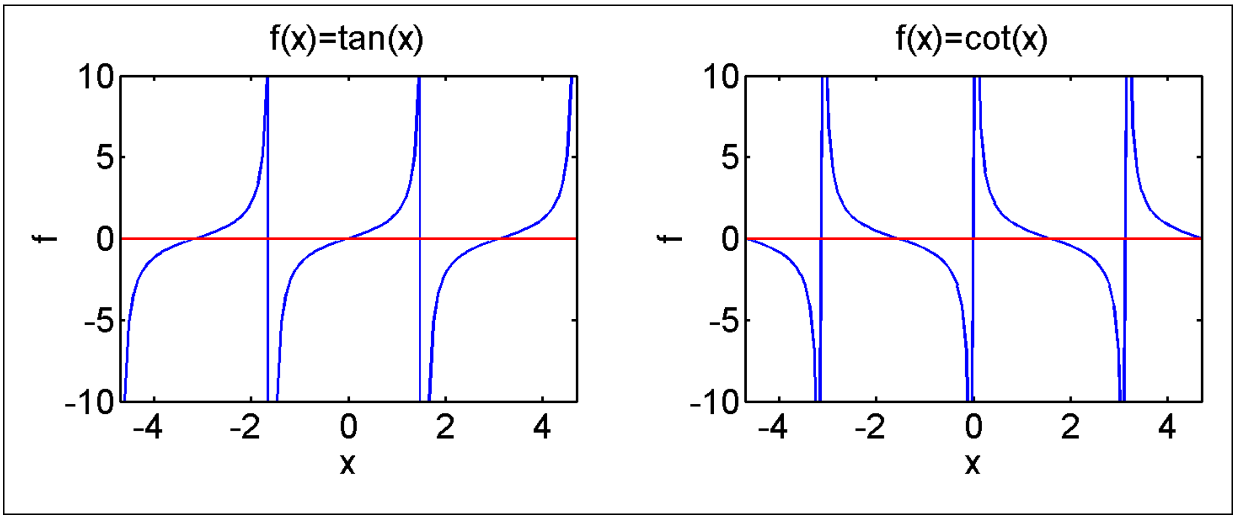
Übungsbeispiel 5 (4)



Übungsbeispiel 5 (5)



Übungsbeispiel 5 (6)



Periodizität: beide Funktionen haben die Periode π

$$\forall n \in \mathbb{Z}: \quad \begin{cases} \tan(x+n\pi) = \tan(x) \\ \cot(x+n\pi) = \cot(x) \end{cases}$$

Symmetrie: beide Funktionen sind antisymmetrisch:

$$\tan(-x) = -\tan(x) \quad (\text{antisymmetrisch})$$
$$\cot(-x) = -\cot(x) \quad (\text{antisymmetrisch})$$

Stetigkeit: die Funktionen sind stetig bis auf

$$\left\{x = \left(n+\tfrac{1}{2}\right)\pi\right\} \quad \forall n \in \mathbb{Z} \quad \text{für tan}$$
$$\{x = n\pi\} \quad \forall n \in \mathbb{Z} \quad \text{für cot}$$

Übungsbeispiel 5 (7)

a)

$$\begin{aligned}
\sin x\cos y+\cos x\sin y &= \frac{1}{2i}\left(e^{ix}-e^{-ix}\right)\frac{1}{2}\left(e^{iy}+e^{-iy}\right)+\frac{1}{2}\left(e^{ix}+e^{-ix}\right)\frac{1}{2i}\left(e^{iy}-e^{-iy}\right)\\
&= \frac{1}{4i}\left(e^{i(x+y)}-e^{-i(x+y)}+e^{i(x-y)}-e^{-i(x-y)}\right)+\frac{1}{4i}\left(e^{i(x+y)}-e^{-i(x+y)}-e^{i(x-y)}+e^{-i(x-y)}\right)\\
&= \frac{1}{2i}\left(e^{i(x+y)}-e^{-i(x+y)}\right)\\
&= \sin\left(x+y\right)
\end{aligned}$$

$$\begin{aligned}
\cos x\cos y-\sin x\sin y &= \frac{1}{2}\left(e^{ix}+e^{-ix}\right)\frac{1}{2}\left(e^{iy}+e^{-iy}\right)-\frac{1}{2i}\left(e^{ix}-e^{-ix}\right)\frac{1}{2i}\left(e^{iy}-e^{-iy}\right)\\
&= \frac{1}{4}\left(e^{i(x+y)}+e^{-i(x+y)}+e^{i(x-y)}+e^{-i(x-y)}\right)+\frac{1}{4}\left(e^{i(x+y)}+e^{-i(x+y)}-e^{i(x-y)}-e^{-i(x-y)}\right)\\
&= \frac{1}{2}\left(e^{i(x+y)}+e^{-i(x+y)}\right)\\
&= \cos\left(x+y\right)
\end{aligned}$$

b)

$$\sin\left(x-x\right)=\sin\left(0\right)=0 \quad \text{und}$$
$$\sin\left(x\right)\underbrace{\cos\left(-x\right)}_{=\cos x}+\cos\left(x\right)\underbrace{\sin\left(-x\right)}_{=-\sin x}=\sin x\cos x-\cos x\sin x=0$$

$$\cos\left(x-x\right)=\cos\left(0\right)=1 \quad \text{und}$$
$$\cos\left(x\right)\underbrace{\cos\left(-x\right)}_{=\cos x}-\sin\left(x\right)\underbrace{\sin\left(-x\right)}_{=-\sin x}=\cos^2 x+\sin^2 x=1$$

Übungsbeispiel 5 (8)

$$e^{\ln x}=x \quad\Rightarrow\quad {}^a\log x={}^a\log\left(e^{\ln x}\right)={}^a\log e\cdot\ln x$$

$$a^{{}^a\log e}=e \quad\Rightarrow\quad \ln a^{{}^a\log e}=\ln e \quad\Rightarrow\quad {}^a\log e\ln a=1 \quad\Rightarrow\quad {}^a\log e=\frac{1}{\ln a}$$

Übungsbeispiel 5 (9)

$$\sinh x=\tfrac{1}{2}\left(e^{x}-e^{-x}\right)$$

$$\sinh\ln f\left(x\right)=\tfrac{1}{2}\left(e^{\ln f(x)}-e^{-\ln f(x)}\right)=\tfrac{1}{2}\left(f\left(x\right)-\frac{1}{f\left(x\right)}\right)\overset{!}{=}x$$

wir bestimmen die Funktion *f(x)*

$$f^2(x) - 2f(x)x - 1 = 0$$
$$f^2(x) - 2f(x)x + x - x - 1 = 0$$
$$\left(f(x) - x\right)^2 = x^2 + 1$$
$$f(x) = x + \sqrt{x^2 + 1}$$

also ist

$$\sinh \ln\left(x + \sqrt{x^2 + 1}\right) = x \quad \text{und daher}$$
$$\mathrm{Arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\cosh x = \tfrac{1}{2}\left(e^x + e^{-x}\right)$$
$$\cosh \ln g(x) = \tfrac{1}{2}\left(e^{\ln g(x)} + e^{-\ln g(x)}\right) = \tfrac{1}{2}\left(g(x) + \frac{1}{g(x)}\right) \stackrel{!}{=} x$$

wir bestimmen die Funktion *g(x)*

$$g^2(x) - 2g(x)x + 1 = 0$$
$$g^2(x) - 2g(x)x + x^2 - x^2 + 1 = 0$$
$$\left(g(x) - x\right)^2 = x^2 - 1$$
$$g(x) = x + \sqrt{x^2 - 1}$$

also ist

$$\cosh \ln\left(x + \sqrt{x^2 - 1}\right) = x \quad \text{und daher}$$
$$\mathrm{Arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$$