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Einladung zur Mathematik

Eine mathematische Einführung und Begleitung zum Studium der Physik und Informatik, Logos Verlag 2002

Lösungen zu den Übungsaufgaben

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Kapitel 6

Übungsbeispiel 6 (1)

$$y = e^{ax}\cos bx$$

$$y' = \left(e^{ax}\right)'\cos bx + e^{ax}\left(\cos bx\right)' = e^{ax}\left(a\cos bx - b\sin bx\right)$$

Übungsbeispiel 6 (2)

$$y = \ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$$

$$\frac{d}{dx}y = \frac{d}{dx}\ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) = \frac{d\ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}{d\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}\frac{d\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}{d\left(\frac{x}{2}+\frac{\pi}{4}\right)}\frac{d\left(\frac{x}{2}+\frac{\pi}{4}\right)}{dx}$$

$$= \frac{1}{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}\left(1+\tan^2\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)\frac{1}{2}$$

$$= \frac{1}{2}\left(\frac{1}{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}+\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)$$

Übungsbeispiel 6 (3)

$$y = x\cos\left(\ln x - \frac{\pi}{4}\right)$$

$$\frac{dy}{dx} = \cos\left(\ln x - \frac{\pi}{4}\right) + x\frac{d\cos\left(\ln x - \frac{\pi}{4}\right)}{dx}$$

$$= \cos\left(\ln x - \frac{\pi}{4}\right) + x\frac{d\cos\left(\ln x - \frac{\pi}{4}\right)}{d\left(\ln x - \frac{\pi}{4}\right)}\frac{d\left(\ln x - \frac{\pi}{4}\right)}{dx}$$

$$= \cos\left(\ln x - \frac{\pi}{4}\right) - x\sin\left(\ln x - \frac{\pi}{4}\right)\frac{1}{x}$$

$$= \cos\left(\ln x - \frac{\pi}{4}\right) - \sin\left(\ln x - \frac{\pi}{4}\right)$$

Übungsbeispiel 6 (4)

$$y = \arctan\frac{a\tan\frac{x}{2} + c}{\sqrt{a^2 - c^2}}$$

$$\frac{dy}{dx} = \frac{d\arctan\frac{a\tan\frac{x}{2}+c}{\sqrt{a^2-c^2}}}{dx} = \frac{d\arctan\frac{a\tan\frac{x}{2}+c}{\sqrt{a^2-c^2}}}{d\,\frac{a\tan\frac{x}{2}+c}{\sqrt{a^2-c^2}}}\frac{1}{\sqrt{a^2-c^2}}\frac{d\left(a\tan\frac{x}{2}+c\right)}{d\,\frac{x}{2}}\frac{1}{2}$$

$$= \frac{1}{1+\left(\frac{a\tan\frac{x}{2}+c}{\sqrt{a^2-c^2}}\right)^2}\frac{a}{\sqrt{a^2-c^2}}\left(1+\tan^2\frac{x}{2}\right)\frac{1}{2}$$

$$= \frac{1}{2}\frac{a^2-c^2}{a^2-c^2+\left(a\tan\frac{x}{2}+c\right)^2}\frac{a}{\sqrt{a^2-c^2}}\left(1+\tan^2\frac{x}{2}\right)$$

$$= \frac{1}{2}\frac{\sqrt{a^2-c^2}\left(1+\tan^2\frac{x}{2}\right)}{a\left(1+\tan^2\frac{x}{2}\right)+2c\tan\frac{x}{2}}$$

Übungsbeispiel 6 (5)

$$y=\frac{a-b\cos x}{a+b\cos x}$$

$$y'=\frac{\left(a-b\cos x\right)'}{a+b\cos x}-\frac{\left(a-b\cos x\right)}{\left(a+b\cos x\right)^2}\left(a+b\cos x\right)'$$

$$=\frac{b\sin x\left(a+b\cos x\right)}{\left(a+b\cos x\right)^2}+\frac{b\sin x\left(a-b\cos x\right)}{\left(a+b\cos x\right)^2}$$

$$=\frac{2ab\sin x}{a+b\cos x}$$

Übungsbeispiel 6 (6)

$$y=\arctan\left(m\tan x\right)$$

$$\frac{dy}{dx}=\frac{d\arctan\left(m\tan x\right)}{dx}=\frac{d\arctan\left(m\tan x\right)}{dm\tan x}\frac{md\tan x}{dx}$$

$$=\frac{1}{1+m^2\tan^2 x}m\left(1+\tan^2 x\right)$$

Übungsbeispiel 6 (7)

$$y=x^{\tan x}$$

$$y'=\frac{dx^{\tan x}}{x}=\frac{dx^{\tan x}}{d\tan x}\frac{d\tan x}{x}=\tan x\cdot x^{\tan x-1}\left(1+\tan^2 x\right)$$

Übungsbeispiel 6 (8)

$$y=\frac{1}{\sqrt{3}}\arctan\frac{x\sqrt{3}}{1-x^2}$$

$$y'=\frac{1}{\sqrt{3}}\frac{d\arctan\frac{x\sqrt{3}}{1-x^2}}{dx}=\frac{1}{\sqrt{3}}\frac{d\arctan\frac{x\sqrt{3}}{1-x^2}}{d\frac{x\sqrt{3}}{1-x^2}}\frac{d\frac{x\sqrt{3}}{1-x^2}}{dx}=\frac{1}{\sqrt{3}}\frac{1}{1+\left(\frac{x\sqrt{3}}{1-x^2}\right)^2}\left(\frac{\sqrt{3}}{1-x^2}-\frac{x\sqrt{3}\left(-2x\right)}{\left(1-x^2\right)^2}\right)$$

$$=\frac{1}{\sqrt{3}}\frac{\left(1-x^2\right)^2}{\left(1-x^2\right)^2+\left(x\sqrt{3}\right)^2}\frac{\sqrt{3}\left(1-x^2\right)+2x^2\sqrt{3}}{\left(1-x^2\right)^2}=\frac{\left(1-x^2\right)+2x^2}{\left(1-x^2\right)^2+3x^2}=\frac{1+x^2}{x^4-2x^2+1+3x^2}$$

$$=\frac{1+x^2}{x^4+x^2+1}$$

Übungsbeispiel 6 (9)

$$y = \frac{a\sin x + b\cos x}{a\sin x - b\cos x}$$

$$y' = \frac{d}{dx}\frac{a\sin x + b\cos x}{a\sin x - b\cos x} = \frac{\left(a\sin x + b\cos x\right)'}{a\sin x - b\cos x} - \frac{\left(a\sin x + b\cos x\right)\left(a\sin x - b\cos x\right)'}{\left(a\sin x - b\cos x\right)^2}$$

$$= \frac{\left(a\sin x - b\cos x\right)\left(a\cos x - b\sin x\right) - \left(a\sin x + b\cos x\right)\left(a\cos x + b\sin x\right)}{\left(a\sin x - b\cos x\right)^2}$$

$$= \frac{\left(a^2 + b^2\right)\sin x\cos x - ab\left(\sin^2 x + \cos^2 x\right) - \left(a^2 + b^2\right)\sin x\cos x - ab\left(\sin^2 x + \cos^2 x\right)}{\left(a\sin x - b\cos x\right)^2}$$

$$= \frac{-2ab}{\left(a\sin x - b\cos x\right)^2}$$

Übungsbeispiel 6 (10)

$$y = \frac{1}{\sqrt{2ax - x^2}}$$

$$y' = \frac{d}{dx}\frac{1}{\sqrt{2ax - x^2}} = -\frac{1}{2}\frac{2a - 2x}{\left(2ax - x^2\right)^{3/2}} = \frac{x - a}{\left(2ax - x^2\right)^{3/2}}$$

Übungsbeispiel 6 (11)

$$y = \frac{\sqrt{2ax + b}}{a}\left(\ln x - 2\right) + \frac{\sqrt{2b}}{a}\left(\ln x - 2\right)$$

$$y' = \frac{d}{dx}\frac{1}{\sqrt{2ax - x^2}} = -\frac{1}{2}\frac{2a - 2x}{\left(2ax - x^2\right)^{3/2}} = \frac{x - a}{\left(2ax - x^2\right)^{3/2}}$$

Übungsbeispiel 6 (17)

$$y = \frac{x}{a + bx}$$

$$y' = \frac{1}{a + bx} - \frac{bx}{\left(a + bx\right)^2} = \frac{a + bx - bx}{\left(a + bx\right)^2} = \frac{a}{\left(a + bx\right)^2}$$

$$y'' = \left(\frac{a}{\left(a + bx\right)^2}\right)' = \frac{-2ab}{\left(a + bx\right)^3}$$

$$y^{(3)} = \left(\frac{-2ab}{\left(a + bx\right)^3}\right)' = \frac{2\cdot 3ab^2}{\left(a + bx\right)^4}$$

$$y^{(4)} = \left(\frac{2\cdot 3ab^2}{\left(a + bx\right)^4}\right)' = \frac{2\cdot 3\cdot 4ab^3}{\left(a + bx\right)^5}$$

Annahme:

$$y^{(n)} = \frac{(-1)^{(n-1)}(n-1)!ab^{(n-1)}}{(a+bx)^n} \quad \Rightarrow$$

$$\left(y^n\right)' = \frac{-nb(-1)^{(n-1)}(n-1)!ab^{(n-1)}}{(a+bx)^{n+1}} = \frac{(-1)^n\, n!ab^n}{(a+bx)^{n+1}} = y^{(n+1)}$$

Übungsbeispiel 6 (18)

$$y = x^2 e^x,$$

$$y' = \left(2x + x^2\right)e^x$$

$$y'' = \left(2 + 2x + 2x + x^2\right)e^x = \left(2 + 4x + x^2\right)e^x$$

$$y^{(3)} = \left(2 + 4x + x^2 + 4 + 2x\right)e^x = \left(6 + 6x + x^2\right)e^x$$

$$y^{(4)} = \left(6 + 6x + x^2 + 6 + 2x\right)e^x = \left(12 + 8x + x^2\right)e^x$$

$$\vdots$$

$$y^{(n)} = \left(0 + 2 + 4 + \ldots 2(n-1) + 2nx + x^2\right)e^x = \left(2\underbrace{\sum_{k=1}^{n-1} k}_{=\frac{n-1}{2}(n-1+1)} + 2nx + x^2\right)e^x$$

$$= \left((n-1)n + 2nx + x^2\right)e^x$$

Übungsbeispiel 6 (19)

$$y = \tan x$$

$$\left.\begin{array}{r} y' = 1 + \tan^2 x \\ y^2 = \tan^2 x \end{array}\right\} \Rightarrow \quad y' = y^2 + \text{Konstante} \quad \Rightarrow \quad y'' = \left(y^2\right)'$$

Übungsbeispiel 6 (20)

Nebenrechnung:

$$(1-x)\left(x^{-n+1} + \ldots + x^{-2} + x^{-1} + 1 + x + x^2 + \ldots + x^{n-1}\right) = \left(x^{-(n-1)} + \ldots + x^{-2} + x^{-1} + 1 + x + x^2 + \ldots + x^{n-1}\right)$$
$$-\left(x^{-n+2} + x^{-n+3} \ldots + x^{-1} + {+1} + x + x^2 + \ldots + x^n\right)$$

$$= x^{-n+1} - x^n$$

$$\Rightarrow \quad x^{-n+1} + \ldots + x^{-2} + x^{-1} + 1 + x + x^2 + \ldots + x^{n-1} = \frac{x^{-n+1} - x^n}{1-x}$$

damit:

$$\frac{1}{2}+\cos x+\cos 2x+\ldots+\cos nx=\frac{1}{2}+\frac{1}{2}\left(e^{ix}+e^{-ix}\right)+\frac{1}{2}\left(e^{i2x}+e^{-i2x}\right)+\ldots+\frac{1}{2}\left(e^{inx}+e^{-inx}\right)$$
$$=\frac{1}{2}\Big\{1+\underset{u}{e^{ix}}+e^{i2x}+\ldots+e^{inx}+e^{-ix}+e^{-i2x}+\ldots+e^{-inx}\Big\}$$
$$=\frac{1}{2}\left\{u^{-n}+u^{-n+1}+\ldots+u^{n-1}+u^{n}\right\}=\frac{1}{2}\frac{u^{-n}-u^{n+1}}{1-u}$$
$$=\frac{1}{2}\frac{\left(u^{-n}-u^{n+1}\right)u^{-1/2}}{\left(1-u\right)u^{-1/2}}=\frac{1}{2}\frac{u^{-n-1/2}-u^{n+1/2}}{u^{-1/2}-u^{1/2}}$$
$$=\frac{1}{2}\frac{\sin\left(n+1/2\right)x}{\sin x/2}$$

ableiten:

$$\frac{1}{2}-\sin x-2\sin 2x-\ldots-n\sin nx=\frac{1}{2}\frac{\left(n+1/2\right)\cos\left(n+1/2\right)x}{\sin x/2}-\frac{1}{4}\frac{\sin\left(n+1/2\right)x}{\left(\sin x/2\right)^{2}}\cos\left(x/2\right)$$

Übungsbeispiel 6 (21)

$$y=\sin 3x-3\sin x,\quad y'=3\left(\cos 3x-\cos x\right)$$

a) Extrema:

$$0=y'=3\left(\cos 3x-\cos x\right)\quad\Rightarrow\quad\cos 3x=\cos x\quad\Rightarrow\quad 3x=x+n\pi$$

Übungsbeispiel 6 (22)

$$y=\frac{x^{2}-1}{x^{2}-x+1}$$

Asymptoten

$$\lim_{x\to\pm\infty}y=\frac{x^{2}}{x^{2}}=1$$

Singularitäten

$$0=x^{2}-x+1=x^{2}-2x\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+1=\left(x-\frac{1}{2}\right)^{2}+\frac{3}{4}$$
$$\Rightarrow\quad x=\frac{1}{2}\pm\sqrt{-\frac{3}{4}}\notin\mathbb{R}$$

Extrema

$$y' = \frac{2x}{x^2 - x + 1} - \frac{x^2 - 1}{\left(x^2 - x + 1\right)^2}\left(2x - 1\right) = \frac{2x\left(x^2 - x + 1\right) - \left(x^2 - 1\right)\left(2x - 1\right)}{\left(x^2 - x + 1\right)^2}$$

$$= \frac{\left(2x^3 - 2x^2 + 2x\right) - \left(2x^3 - 2x - x^2 + 1\right)}{\left(x^2 - x + 1\right)^2} = \frac{-x^2 + 4x - 1}{\left(x^2 - x + 1\right)^2}$$

$$y' = 0 \quad \Rightarrow \quad x^2 - 4x + 1 = x^2 - 2x2 + 4 - 4 + 1 = \left(x - 2\right)^2 - 3 = 0 \qquad \text{also}$$

$$x = 2 \pm \sqrt{3} = \begin{cases} 3.7321 \\ 0.2679 \end{cases}$$

Zweite Ableitung

$$y'' = \left(\frac{-x^2 + 4x - 1}{\left(x^2 - x + 1\right)^2}\right)' = \frac{-2x + 4}{\left(x^2 - x + 1\right)^2} - \frac{-x^2 + 4x - 1}{\left(x^2 - x + 1\right)^3}\frac{2x - 1}{2}$$

$$= \frac{2\left(x^2 - x + 1\right)\left(-2x + 4\right) - \left(-x^2 + 4x - 1\right)\left(2x - 1\right)}{2\left(x^2 - x + 1\right)^3}$$

$$= \frac{\left(-4x^3 + 12x^2 - 12x + 8\right) - \left(-2x^3 + 8x^2 - 2x + x^2 - 4x + 1\right)}{2\left(x^2 - x + 1\right)^3}$$

$$= \frac{-2x^3 + 3x^2 - 6x + 7}{2\left(x^2 - x + 1\right)^3}$$

Nullstellen der zweiten Ableitung

$$y'' = 0 \quad \Rightarrow \quad -2x^3 + 3x^2 - 6x + 7 = \left(x - \frac{4}{5}\right)\left(ax^2 + bx + c\right)$$

$$= ax^3 + \left(b - \frac{4a}{5}\right)x^2 + \left(c - \frac{4b}{5}\right)x - \frac{4c}{5} \qquad \text{also}$$

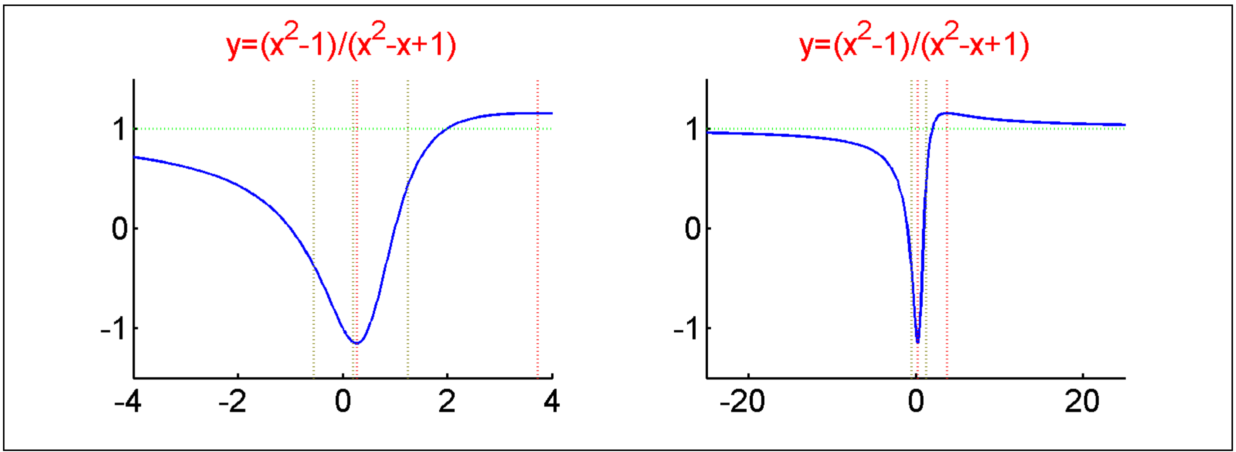
$$\begin{cases} a = -2 \\ b - \frac{4a}{5} = b + \frac{8}{5} = 3 \quad \Rightarrow \quad b = \frac{15 - 8}{5} = \frac{7}{5} \\ c - \frac{4b}{5} = c - \frac{28}{25} = -6 \quad \Rightarrow \quad c = \frac{28 - 150}{25} = -\frac{132}{25} \cong 5.28 \\ -\frac{4c}{5} = 7 \quad \Rightarrow \quad c = \frac{28}{5} \cong 5.6 \end{cases}$$

$$0 = ax^2 + bx + c = x^2 + bx/a + c/a = x^2 + 2x\frac{b}{2a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}$$

also

$$x=-\frac{b}{2a}\pm\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}=-\frac{7}{2\cdot5\cdot(-2)}\pm\sqrt{\left(\frac{7}{2\cdot5\cdot(-2)}\right)^2-\frac{28}{5\cdot(-2)}}=\frac{7}{20}\pm\sqrt{\frac{49}{400}+\frac{14}{5}}=0.35\pm0.9$$

$$x\cong\begin{cases}-0.55\\ 0.2\\ 1.25\end{cases}$$



Übungsbeispiel 6 (23)

Wiensches Verschieungsgesetz: Das Maximum der Strahlungsdichte eines schwarzen Körpers verschiebt sich mit wachsenden Temperaturen zu kleineren Wellenlängen.

$$\rho=\frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{e^{\hbar\omega/kT}-1}=\frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{e^{2\pi\hbar/k\lambda T}-1}$$

Der Ausdruck hängt hinsichtlich der Temperatur nur vom Produkt λT ab. Solange dieser Ausdruck konstant ist (Wiensche Verschiebungsgesetz) ändert sich die spektrale Energieverteilung nicht, also verschiebt sich auch das Maximum der Verteilung nicht.

Übungsbeispiel 6 (24)

a)

$$x_1=rx_0$$
$$x_2=rx_1=r^2x_0$$
$$\vdots$$
$$x_N=rx_{N-1}=r^Nx_0=e^{\ln r^N}x_0=e^{\frac{N\ln r}{a}}x_0=e^{aN}x_0$$

b)

$$N = \frac{\text{Seefläche/2}}{\text{Blattfläche}} = \frac{200 \cdot 100 \cdot km^2 / 2}{10 \cdot 10 \cdot cm^2} = \frac{100 \cdot (100.000cm)^2}{1 \cdot cm^2}$$
$$= 10^{12} = 1000 \cdot 1000 \cdot 1000 \cdot 1000 \cong \left(2^{10}\right)^4 = 2^{40}$$

$$x_0 = 1$$
$$x_1 = 2$$
$$x_2 = 4 = 2^2$$
$$x_3 = 8 = 2^3$$
$$\vdots$$
$$x_k = 2^k = 10^{12} \cong 2^{40}$$

es dauert also 40 Monate (gute 3 Jahre) bis der See zugewachsen ist

c) der Rest des Sees ist in einem weiteren Monat zugewachsen

d) Wären genug Nährstoffe und Lebensraum vorhanden, dann nähme die Bevölkerung am Schluss explosionsartig zu