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Einladung zur Mathematik

Eine mathematische Einführung und Begleitung zum Studium der Physik und Informatik, Logos Verlag 2002

Lösungen zu den Übungsaufgaben

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Kapitel 9

Übungsbeispiel 9.5.1

a)

$$f(x,y) = 2x^2 + 3y^2$$

$$\frac{\partial f}{\partial x} = 4x, \quad \frac{\partial f}{\partial y} = 6y, \quad \frac{\partial^2 f}{\partial x^2} = 4, \quad \frac{\partial^2 f}{\partial y^2} = 6, \quad \frac{\partial^2 f}{\partial x \partial y} = 0;$$

b)

$$f(x,y) = \sin(x+3y)$$

$$\frac{\partial f}{\partial x} = \cos(x+3y), \quad \frac{\partial f}{\partial y} = 3\cos(x+3y),$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x+3y), \quad \frac{\partial^2 f}{\partial y^2} = -9\sin(x+3y), \quad \frac{\partial^2 f}{\partial x \partial y} = -3\sin(x+3y);$$

c)

$$f(x,y) = e^{\left(x^2+y^2\right)}$$

$$\frac{\partial f}{\partial x} = 2xe^{\left(x^2+y^2\right)}, \quad \frac{\partial f}{\partial y} = 2ye^{\left(x^2+y^2\right)},$$

$$\frac{\partial^2 f}{\partial x^2} = \left(2+4x^2\right)e^{\left(x^2+y^2\right)}, \quad \frac{\partial^2 f}{\partial y^2} = \left(2+4y^2\right)e^{\left(x^2+y^2\right)}, \quad \frac{\partial^2 f}{\partial x \partial y} = 4xye^{\left(x^2+y^2\right)};$$

d)

$$f(x,y) = \log\left[x^2 \sin^2(x+y)\right]$$

$$\frac{\partial f}{\partial x} = \frac{2\sin(x+y) + 2x\cos(x+y)}{x\sin(x+y)} = 2x^{-1} + 2\cot(x+y),$$

$$\frac{\partial f}{\partial y} = \frac{2\cos(x+y)}{\sin(x+y)} = 2\cot(x+y),$$

$$\frac{\partial^2 f}{\partial x^2} = -2x^{-2} - 2\frac{1}{\sin^2(x+y)}, \quad \frac{\partial^2 f}{\partial y^2} = -2\frac{1}{\sin^2(x+y)},$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2\frac{1}{\sin^2(x+y)};$$

Übungsbeispiel 9.5.2

$$f(x,y) = xy\frac{x^2 - y^2}{x^2 + y^2} = \frac{x^3 y - xy^3}{x^2 + y^2}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{3x^2 y - y^3}{x^2 + y^2} - \frac{\left(x^3 y - xy^3\right)2x}{\left(x^2 + y^2\right)^2} = \frac{\left(3x^2 y - y^3\right)\left(x^2 + y^2\right)}{\left(x^2 + y^2\right)^2} + \frac{-2x^4 y + 2x^2 y^3}{\left(x^2 + y^2\right)^2}$$

$$= \frac{(3-2)x^4 y + (3-1+2)x^2 y^3 - y^5}{\left(x^2 + y^2\right)^2}$$

$$= \frac{x^4 y + 4x^2 y^3 - y^5}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{x^4 + 12x^2 y^2 - 5y^4}{\left(x^2 + y^2\right)^2} - 2\frac{\left(x^4 y + 4x^2 y^3 - y^5\right)2y}{\left(x^2 + y^2\right)^3}$$

$$= \frac{\left(x^4 + 12x^2 y^2 - 5y^4\right)\left(x^2 + y^2\right)}{\left(x^2 + y^2\right)^3} + \frac{-4x^4 y^2 - 16x^2 y^4 + 4y^6}{\left(x^2 + y^2\right)^3}$$

$$= \frac{x^6 + (1+12-4)x^4 y^2 + (12-5-16)x^2 y^4 + (-5+4)y^6}{\left(x^2 + y^2\right)^3}$$

$$= \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{\left(x^2 + y^2\right)^3}$$

f und die gemischte Ableitung zweiter Ordnung sind antisymmetrische in *x* und *y*:

$$f(x,y) = -f(y,x), \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = -\frac{\partial^2 f}{\partial x \partial y}(y,x)$$

Übungsbeispiel 9.5.3

$$f\left(x,y,z\right)=x^2+y^2+xz$$

$$xf_x+yf_y+zf_z=x\left(2x+z\right)+y(2y)+z\left(x\right)=2x^2+2y^2+2xz=2f$$

Übungsbeispiel 9.5.4

$$f\left(\lambda x\right)=\lambda^{\mu}f\left(x\right)$$

Potenzreihenentwicklung

$$f\left(x\right)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}c_{k_1\ldots k_n}x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$$

$$x\cdot\nabla f\left(x\right)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}x_1k_1c_{k_1\ldots k_n}x_1^{k_1-1}x_2^{k_2}\ldots x_n^{k_n}+\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}x_2k_2c_{k_1\ldots k_n}x_1^{k_1}x_2^{k_2-1}\ldots x_n^{k_n}$$

$$+\ldots\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}x_nk_nc_{k_1\ldots k_n}x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n-1}$$

$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}k_1c_{k_1\ldots k_n}x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}+\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}k_2c_{k_1\ldots k_n}x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$$

$$+\ldots\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}k_nc_{k_1\ldots k_n}x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$$

$$\underbrace{=}_{x\cdot\nabla f(x)=\mu f(x)}\mu\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}c_{k_1\ldots k_n}x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$$

$$\Rightarrow\quad c_{k_1\ldots k_n}\begin{cases}\neq 0 & \text{wenn}\sum_{\mathrm{i=1}}^{\mathrm{n}}k_i=\mu\\ =0 & \text{sonst}\end{cases}$$

dann ist

$$f\left(\lambda x\right)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}c_{k_1\ldots k_n}\lambda^{\sum k_i}x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}=\lambda^{\mu}f\left(x\right)$$