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Einladung zur Mathematik

Eine mathematische Einführung und Begleitung zum Studium der Physik und Informatik, Logos Verlag 2002

Lösungen zu den Übungsaufgaben

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Kapitel 12

Übungsbeispiel 12.4.1

a)

$$\int_0^2\int_0^3 xy(x-y)\,dydx = \int_0^2\int_0^3 \left(x^2y - xy^2\right)dydx = \int_0^2 \left[\tfrac{1}{2}x^2y^2 - \tfrac{1}{3}xy^3\right]_0^3 dx$$

$$= \int_0^2 \left(\tfrac{9}{2}x^2 - \tfrac{27}{3}x\right)dx = \left[\tfrac{9}{2}\tfrac{1}{3}x^3 - \tfrac{27}{3}\tfrac{1}{2}x^2\right]_0^2 = \tfrac{9}{2}\tfrac{8}{3} - \tfrac{27}{3}\tfrac{4}{2} = 12 - 18$$

$$= -6$$

b)

$$\int_1^2\int_0^y x\,dxdy = \int_1^2\int_0^y \left[\tfrac{1}{2}x^2\right]_0^y dy = \int_1^2\int_0^y \tfrac{1}{2}y^2 dy = \left[\tfrac{1}{2}\tfrac{1}{3}y^3\right]_1^2 = \tfrac{1}{6}(8-1) = \tfrac{7}{6}$$

c)

$$\int_0^1\int_{\sqrt{y}}^{2-y} y^2\,dxdy = \int_1^2 \left[y^2x\right]_{\sqrt{y}}^{2-y} dy = \int_1^2 \left(2 - y - \sqrt{y}\right)y^2 dy = \int_1^2 \left(2y^2 - y^3 - y^{5/2}\right)dy$$

$$= \left[\tfrac{2}{3}y^3 - \tfrac{1}{4}y^4 - \tfrac{2}{7}y^{7/2}\right]_1^2 = \left(\tfrac{2}{3}8 - \tfrac{1}{4}16 - \tfrac{2}{7}32\right) - \left(\tfrac{2}{3} - \tfrac{1}{4} - \tfrac{2}{7}\right)$$

$$= \tfrac{16-2}{3} - 4 + \tfrac{1}{4} - \tfrac{64-2}{7} = \tfrac{14}{3} - 4 + \tfrac{1}{4} - \tfrac{62}{7} = \tfrac{14-12}{3} - \tfrac{62}{7} + \tfrac{1}{4} = \tfrac{2}{3} - \tfrac{62}{7} + \tfrac{1}{4}$$

$$= \tfrac{8+3}{12} - \tfrac{62}{7} = \tfrac{11}{12} - \tfrac{62}{7} = \tfrac{77-744}{84} = -\tfrac{667}{84}$$

Übungsbeispiel 12.4.2

$$\text{Ellipse: } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \Rightarrow \quad y_{\max}=\frac{b}{a}\sqrt{a^2-x^2}$$

$$\int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dydx = 2\int_{-a}^{a}\left[y\right]_{0}^{\frac{b}{a}\sqrt{a^2-x^2}}dx = 4\frac{b}{a}\int_{0}^{a}\sqrt{a^2-x^2}dx = 4\frac{b}{a}\left[\frac{1}{2}x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}\right]_0^a$$

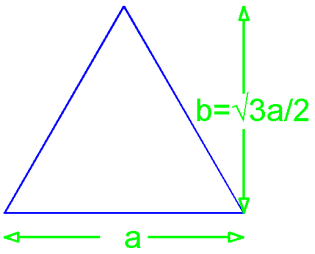
$$= 4\frac{b}{a}\left[\left(0+a^2\arcsin 1\right)-\left(0+a^2\arcsin 0\right)\right] = 4\frac{b}{a}\left(a^2\frac{\pi}{4}\right)$$

$$= ab\pi$$

Übungsbeispiel 12.4.3

$$a^2=\left(\frac{a}{2}\right)^2+b^2 \quad \Rightarrow \quad b=\sqrt{a-\frac{a}{4}}=\frac{\sqrt{3}}{2}a$$

$$y(x)=\sqrt{3}\left(\frac{a}{2}-x\right) \text{ für } x\in\left[0,\frac{a}{2}\right]$$



$$F = 2\int_0^{a/2}\int_0^{y(x)} dy'dx = 2\int_0^{a/2}\left[y'\right]_0^{y(x)}dx = 2\int_0^{a/2}\sqrt{3}\left(\frac{a}{2}-x\right)dx = 2\sqrt{3}\left[\frac{a}{2}x-\frac{1}{2}x^2\right]_0^{a/2}$$

$$= 2\sqrt{3}\left(\left(\frac{a}{2}\right)^2-\frac{1}{2}\left(\frac{a}{2}\right)^2\right) = 2\sqrt{3}\frac{1}{2}\left(\frac{a}{2}\right)^2 = \frac{\sqrt{3}}{4}a^2$$

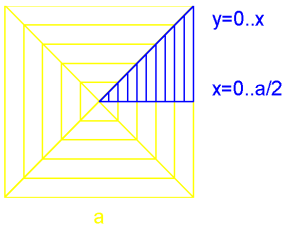
Übungsbeispiel 12.4.4

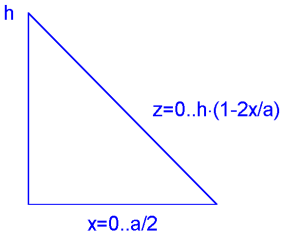
$$SP=\int_0^a\int_0^{bx/a}\begin{pmatrix}x\\y\end{pmatrix}dydx=\int_0^a\int_0^{bx/a}\left[\begin{pmatrix}xy\\y^2/2\end{pmatrix}\right]_0^{bx/a}dx=\int_0^a\begin{pmatrix}bx^2/a\\b^2x^2/2a^2\end{pmatrix}dx=\left[\begin{pmatrix}bx^3/3a\\b^2x^3/6a^2\end{pmatrix}\right]_0^a$$

$$=\frac{1}{3}\begin{pmatrix}ba^3/a\\b^2a^3/2a^2\end{pmatrix}=\frac{ab}{3}\begin{pmatrix}a\\b/2\end{pmatrix}$$

Übungsbeispiel 12.4.6

$$x\in\left[0,\frac{a}{2}\right];\quad y(x)=x;\quad z(x)=h\left(1-\frac{2x}{a}\right)$$





$$M = 8\int_0^{a/2}\int_0^{x}\int_0^{z(x)} \rho\, dz\, dy\, dx = 8\rho h \int_0^{a/2}\int_0^{x}\left(1-\frac{2x}{a}\right) dy\, dx = 8\rho h \int_0^{a/2} x\left(1-\frac{2x}{a}\right) dx$$

$$= 8\rho h\left[\frac{x^2}{2}-\frac{2x^3}{3a}\right]_0^{a/2} = 8\rho h\left(\frac{a^2}{8}-\frac{2a^2}{3\cdot 8}\right) = \rho h\left(\frac{3a^2}{3}-\frac{2a^2}{3}\right)$$

$$= \frac{1}{3}a^2\rho h$$