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Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

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1.1 Vectors and Metric

A first step to theoretical physics is the mathematical description of space and time. This is based on coordinate systems consisting of an origin, basis vectors and coordinates.

The basis vectors determine directions or axis in space, the coordinates are the distance along these axes from the origin to a space point.

First, there is the space vector with three basis vectors and coordinates for *x, y* and *z*. To take the invariance of the speed of light into account, a fourth component is needed. In the case of the spacetime vector, this component is the product of the imaginary unit *i*, the speed of light *c,* and time *t.*

$$\text{Space vector:} \quad \vec{r} = \sum_{i=1}^{3} \underbrace{x_i}_{\text{Coordinate}} \underbrace{\hat{e}_i}_{\text{Basis}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{Spacetime vector:} \quad \vec{r}^{(4)} = \sum_{\mu=1}^{4} x_\mu \hat{e}_\mu = \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix}$$

(1-1)

Invariance

Physical laws are objective or context independent, their formulation cannot depend on the arbitrary choice of a coordinate system. The representation of a vector, however, depends on such a choice. Independent or invariant or objective is the length of a vector, the scalar product of two vectors respectively the angle between them, or the scalar triple product of three vectors.

$$\text{Length of a vector:} \quad a^2 \equiv \vec{a} \cdot \vec{a} = \sum_{i,j=1}^{3} a_i \underbrace{\hat{e}_i \cdot \hat{e}_j}_{\delta_{ij}} a_j = \sum_{i=1}^{3} a_i^2$$

$$\text{Scalar produkt respectively angle } \alpha: \quad \vec{a} \cdot \vec{b} = ab\cos\alpha = \sum_{i=1}^{3} a_i b_i$$

$$\text{Scalar triple product:} \quad V = \det\left(\vec{a}, \vec{b}, \vec{c}\right) = \vec{a} \times \vec{b} \cdot \vec{c}$$

(1-2)

Generalized coordinates

By choosing a suitable coordinate system, physical problems can be simplified or even solved. Therefore, we consider generalized coordinates, their basis vectors and metric. We do not normalize the basis vectors and call them ground vectors. This way, we can treat non-orthogonal coordinates and perform calculations on curved subspaces, for example two-dimensional planes in 3-space.

$$\text{Generalized coordinates:} \quad q^i: \quad x_i \to x_i\left(q^1,\ldots,q^N\right)$$

$$\text{Ground vectors:} \quad \vec{g}_i \equiv \frac{\partial \vec{r}}{\partial q^i} \quad (\text{not normalized basis vectors!})$$

$$\text{Metric:} \quad g_{ij} \equiv \vec{g}_i \cdot \vec{g}_j = \begin{cases} \delta_{ij} & \text{for carthesian coordinates} \\ g_i^2 \delta_{ij} & \text{for } \underbrace{\text{orthogonale coordinates}}_{\text{Cylinder c., Spherical c.}} \\ g_{ij} \neq 0 & \text{general} \end{cases}$$

$$\text{Coordinate lines:} \quad \vec{r}_i = \vec{r}\left(q^i\right) \quad \textit{with} \quad q^j = \text{constant} \ \forall\, j \neq i$$

(1-3)

Example (for orthogonal coordinates):

$$\text{Cylindrical coordinates:} \quad \vec{r} = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix}$$

$$\text{Ground vectors:} \quad \vec{g}_\rho = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \quad \vec{g}_\varphi = \begin{pmatrix} -\rho\sin\varphi \\ \rho\cos\varphi \\ 0 \end{pmatrix}, \quad \vec{g}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Metric:} \quad G \equiv \left(g_{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1-4)

Contravariant ground vectors

The scalar product of two vectors is based on the orthogonality of left and right ground vectors. For non-orthogonal coordinates, a distinction must be made between covariant (right) and contravariant (left) ground vectors. A contravariant ground vector can be constructed as gradient of the corresponding generalized coordinate, since this gradient is orthogonal to the level surface of the other ground vectors.

$$\text{Vector:} \qquad \vec r = \underbrace{x^i \vec g_i}_{\substack{\text{Einstein's} \\ \text{sum convention}}}$$

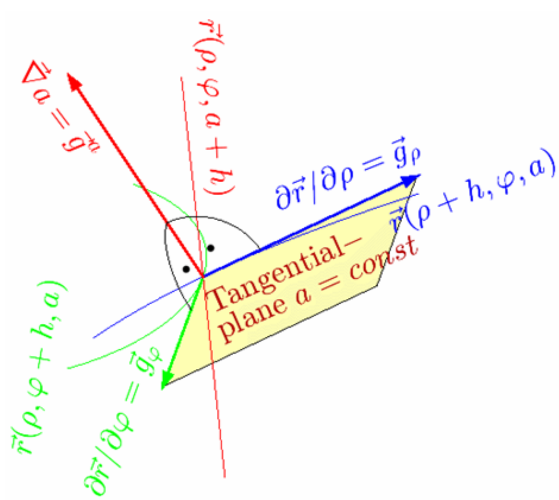
$$\text{Scalar produkt:} \qquad \vec r^{\,2} = \underbrace{x_i \vec g^i}_{\text{contravariant}} \cdot \underbrace{x^j \vec g_j}_{\text{covariant}} = x_i \underbrace{\vec g^i \cdot \vec g_j}_{\stackrel{!}{=}\delta_i^j} x^j = x_i x^i$$

$$\text{Contravariant ground vector:} \qquad \vec g^i \equiv \vec\nabla q^i \left(x_1, x_2, x_3\right)$$

$$\text{Orthonormality:} \qquad \vec g^i \cdot \vec g_j = \vec\nabla q^i \cdot \frac{\partial \vec r}{\partial q^j} = \left(\sum_{k=1}^{3} \frac{\partial q^i\left(x_1,x_2,x_3\right)}{\partial x_k}\hat e_k\right)\cdot\left(\sum_{l=1}^{N}\frac{\partial x_l\left(q^1,\ldots,q^N\right)}{\partial q^j}\hat e_l\right)$$

$$= \sum_{k=1}^{3} \frac{\partial q^i\left(x_1,x_2,x_3\right)}{\partial x_k}\frac{\partial x_k\left(q^1,\ldots,q^N\right)}{\partial q^j} = \frac{\partial q^i}{\partial q^j} = \delta_j^i$$

(1-5)



Construction of a contravariant ground vector. The ground vectors of r and φ span the plane with a=constant. The gradient of $a(x_i)$ *is orthogonal to this plane and therefore also to the relevant covariant ground vectors.*

Covariant and contravariant expansion

Vectors can be represented with both systems of ground vectors $\vec{g_i}$ and $\vec{g^i}$. We consider the scalar product of a vector with a ground vector, use the orthonormality of the ground vectors, and compare the coefficients.

$$\text{Scalar product:} \qquad \vec a \cdot \vec g^i = \underbrace{\left(\vec a \cdot \vec g^k\right)\delta_k^i}_{\text{Sum convention!}} = \left(\vec a \cdot \vec g^k\right)\vec g_k \cdot \vec g^i$$

$$\text{Comparison of coefficients in } \vec g^i\text{:} \qquad \vec a = \left(\vec a \cdot \vec g^k\right)\vec g_k = a^k \vec g_k$$

$$\text{Analogous:} \qquad \vec a = \left(\vec a \cdot \vec g_k\right)\vec g^k = a_k \vec g^k$$

(1-6)

Metric und scalar product

The metric of the contravariant ground vectors can be calculated by inversion of the covariant metric. Therefore, contravariant ground vectors are not explicitly required in the rest of this script.

Orthogonality: $\delta_i^j = \vec{g}_i \cdot \vec{g}^j = \vec{g}_i \cdot \left(\vec{g}^j \cdot \vec{g}^k\right)\vec{g}_k = g^{jk} g_{ki} \quad \Rightarrow$

Contravariant metric: $\boxed{G_i \equiv \left(g^{ik}\right) = G^{-1}}$

Scalar product: $\vec{a} \cdot \vec{b} = \vec{g}^i a_i \cdot \vec{g}^j b_j = g^{ij} a_i b_j = g_{ij} a^i b^j = a_i b^j$

Length square of a vector: $a^2 = g^{ij} a_i a_j = g_{ij} a^i a^j$

(1-7)

Raising and lowering of indices

The metric allows to change between covariant and contravariant ground vectors and coordinates. This is done by raising and lowering of indices.

Raising and lowerding of indices: $\overbrace{a^i \vec{g}_i}^{\vec{r}} = \overbrace{a_j \vec{g}^j}^{\vec{r}} = a_j \underbrace{\left(\vec{g}^j \cdot \vec{g}^i\right)\vec{g}_i}_{\text{Representation of a contravariant basis vector by a covariant}} = \underbrace{a_j g^{ji}}_{\substack{=a^i \\ \text{(Comparison of coefficients in } \vec{g}_i)}} \vec{g}_i$

or: $a^i = g^{ij} a_j, \quad a_i = g_{ij} a^j$

(1-8)

Total differential and gradient

The differential of the space vector is directly connected to the definition of the ground vectors. The differential of a scalar field leads to the gradient of the field. It becomes clear that the space vector is covariant, and that the field gradient is contravariant.

Differential of a space vector: $d\vec{r} \equiv \frac{\partial \vec{r}}{\partial q^i} dq^i = \vec{g}_i dq^i$

Differential of a field S: $dS = \frac{\partial S}{\partial q^i} dq^i = \frac{\partial S}{\partial q^j} \delta_i^j dq^i = \underbrace{\frac{\partial S}{\partial q^j} \vec{g}^j}_{\equiv \vec{\nabla} S} \cdot \vec{g}_i dq^i = \vec{\nabla} S \cdot d\vec{r}$

Gradient in generalized coordinates: $\boxed{\vec{\nabla} S = \underbrace{\frac{\partial S}{\partial q^j}}_{\equiv p_j} \vec{g}^j = p_j \vec{g}^j}$

(1-9)

This brings us to the central point of theoretical physics. The scalar product of the space vector and the momentum vector defines a scalar quantity, the action *S.* The so-called canonical momentum is the gradient of the action, and is thus contravariant. Relativity of space and time lead to four-space and four-momentum, therefore the action includes contributions of space and time .

Momentum	Gradient of action	Canonical momenta p_i	Kinetic energy
$\vec{p}$	$\vec{p} = \vec{\nabla} S$	$\vec{p} = p_i \vec{g}^i$	$T = \frac{1}{2m}\vec{p}^2 = g^{ij} \underbrace{p_i p_j}_{\substack{\text{components} \\ \text{of a gradient} \\ \to \text{contravariant}}}$

(1-10)

Proceeding

Once appropriate coordinates have been chosen, the Jacobian can be derived. Its rows are the ground vectors. From the Jacobi matrix, the metric is obtained through matrix multiplication and the contravariant metric through matrix inversion.

$$
\begin{aligned}
&\text{Coordinates:} && x^i\left(q^1,\dots,q^N\right) \\
&\text{Jacobian:} && J = \left(\frac{\partial x^i}{\partial q^j}\right) = \left(\frac{\partial \vec{r}}{\partial q^j}\right) = \left(\vec{g}_j\right) \\
&\text{Metric:} && G = \left(\vec{g}_i \cdot \vec{g}_j\right) = J^T J, \quad G_i = \left(g_{ij}\right)^{-1} \\
&\text{Velocity} \sim \text{momentum:} && m\dot{q}^i = g^{ij} p_j \\
&\text{Kinetic energie:} && T = \frac{1}{2m}\vec{p}^2 = \frac{1}{2m} g^{ij} p_i p_j
\end{aligned}
$$

(1-11)

Example: Cylindrical coordinates

$$
\begin{aligned}
&\text{Coordinates:} && \vec{r} = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix} \\
&\text{Jacobian:} && J = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left(\vec{g}_\rho, \vec{g}_\varphi, \vec{g}_z\right) \\
&\text{Metric:} && G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_i = G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\text{Velocity} \sim \text{Momentum:} && \begin{cases} m\dot{\rho} = p_\rho \\ m\dot{\varphi} = \rho^{-2} p_\varphi \\ m\dot{z} = p_z \end{cases} \\
&\text{Kinetic energie:} && T = \frac{1}{2m}\left(p_\rho^2 + \frac{1}{\rho^2} p_\varphi^2 + p_z^2\right)
\end{aligned}
$$

(1-12)