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Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

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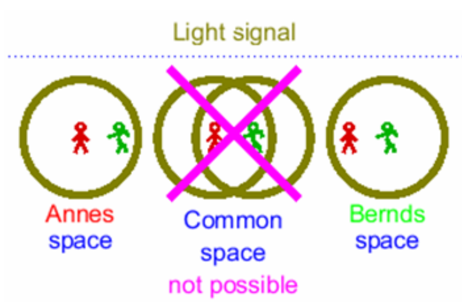
2.1 Structure of Physical Spacetime

In contrast to water waves or sound waves, for light signals there is no medium to which the velocity of the signals can relate. There is only the observer, and nothing else. Light signals propagate relative to the observer.

The problem becomes obvious when two observers, perhaps Anne and Bernd, look at the same light signal. The signal propagates in all directions at the same speed *c*. That applies to both, even if they move relative to each other!



And that is not possible in a common, objective space. They both see themselves at the centre of a sphere formed by the propagating light signal, even if they move away from each other! But since every sphere has only a single, unique centre, this is not possible in an objective, common space for both. The light propagation takes place for each of them in an individual space. Space exists in relation to the observer, for each observer.

Propagation of light signals:	no medium like water or air
Consequence:	*c* is independent of the reference system
and:	spacetime only exists in relation to an observer
Philosophy, psychology, art, ...:	Space and time arise in the consciousness of the observer

(2-1)

Even in a computer-generated virtual reality, the space seems absolute to us. We immediately feel like we exist in this space and often quickly forget that this is not our 'real' space. Here it becomes obvious, that in our consciousness, relative space appears absolute.

Light propagation can be expressed by the length of a vector, the length of four-position. This length is independent of the reference system or invariant, it is the same for both Anne and Bernd. This allows us to determine the transformation between the reference systems, the Lorentz transformation.

$$\text{Propagation of light signals:} \quad \underbrace{x^2+y^2+z^2}_{\text{Squared distance from light source}} - \underbrace{c^2t^2}_{\text{Squared distance in time } t} = 0$$

$$\text{In a different coordinate system:} \quad x'^2+y'^2+z'^2-c^2t'^2=0, \quad (c=c'!)$$

$$\text{Invariant formulation as length of vectors:} \quad \left(\vec{r}^{(4)}\right)^2 = \left(\vec{r}'^{(4)}\right)^2 = 0$$

$$\text{Here the four-position is:} \quad \vec{r}^{(4)} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z + ict\hat{e}_t = \begin{pmatrix} \vec{r} \\ ict \end{pmatrix}$$

(2-2)

Minkowski metric

The imaginary unit *i* in the 4th component of the four-position could give way to esoteric speculation. It can be easily eliminated by a suitable choice of generalized coordinates.

$$\text{Generalized coordinates:} \quad q^i = x^i, \quad q^4 = ct$$

$$\text{Differential of four-space:} \quad d\vec{r}^{(4)} = \vec{g}_i^{(4)} dq^i + \vec{g}_4^{(4)} dq^4, \quad i \in \{1..3\}$$

$$\text{Ground vectors:} \quad \vec{g}_i^{(4)} = \hat{e}_i, \quad \vec{g}_4^{(4)} = \frac{\partial \vec{r}^{(4)}}{\partial ct} = i\hat{e}_4$$

$$\text{Contravariant:} \quad \vec{g}^{(4)i} = \hat{e}_i, \quad \underbrace{\vec{g}^{(4)4} = -i\hat{e}_4}_{\vec{g}_\mu^{(4)} \cdot \vec{g}^{(4)\nu} = \delta_\mu^\nu}$$

$$\text{Minkowski metric:} \quad (g_{\mu\nu}) = (g^{\mu\nu}) = diag(1,1,1,-1)$$

(2-3)

Lorentz transformation (LT)

The Lorentz transformation mediates between the various observers or their reference systems.

$$\text{Reference systems:} \quad K, \quad K'$$

$$\text{Relative velocity:} \quad v$$

$$\text{Lorentz-transformation:} \quad \begin{pmatrix} x' \\ ct' \end{pmatrix} = \underbrace{T_L}_{\substack{\mathbb{R}^2 \times \mathbb{R}^2 \\ \text{4 elements}}} \begin{pmatrix} x \\ ct \end{pmatrix}, \quad \underbrace{y'=y, z'=z}_{\text{by suitable choice of the spatial directions}}$$

(2-4)

This transformation is determined by the condition that the length of the four-space and the velocity of light c are the same in each system. This gives four coupled quadratic equations. Their solution defines the Lorentz transformation.

$$\text{Invariance condition:} \qquad \underbrace{x'^2-(ct')^2}_{(a_{11}x+a_{12}ct)^2-(a_{21}x+a_{22}ct)^2} = x^2-(ct)^2$$

$$\text{Comparison of coefficients:}$$

$$\underbrace{\left(a_{11}^2-a_{21}^2-1\right)}_{=0}x^2+\underbrace{\left(a_{12}^2-a_{22}^2-1\right)}_{=0}(ct)^2+2\underbrace{\left(a_{11}^2a_{12}^2-a_{21}^2a_{22}^2\right)}_{=0}x\cdot ct=0$$

$$\text{Relative motion:} \quad \underbrace{0=x'}_{\text{Rest frame}}=a_{11}x+a_{12}ct \quad \text{and} \quad x=v\cdot t$$

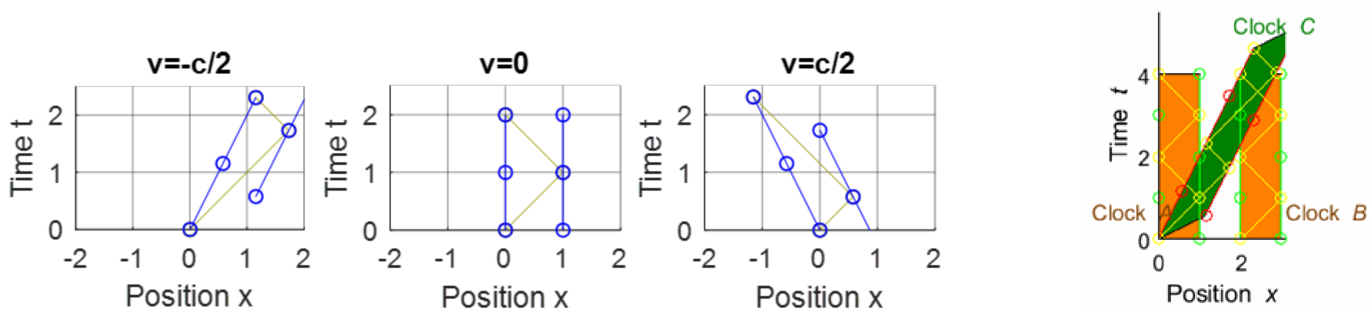
$$\text{Solution:} \qquad T_L=\begin{pmatrix} \frac{c}{\sqrt{c^2-v^2}} & \frac{-v}{\sqrt{c^2-v^2}} \\ \frac{-v}{\sqrt{c^2-v^2}} & \frac{c}{\sqrt{c^2-v^2}} \end{pmatrix} = \underbrace{\gamma}_{\sqrt{\frac{1}{1-v^2/c^2}}}\begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix}$$

(2-5)

Light clocks

A light clock consists of two mirrors, between which a light signal is reflected. A unit of time can then be defined as the time it takes light to travel from one mirror to the other. With the Lorentz transformation, the world lines of a light clock can be transformed into reference frames with different relative velocities, the speed of light c remaining the same. In a frame that moves relative to the light clock, this time appears longer.

Experimentally, this can be demonstrated with three light clocks, two stationary and one in motion. At the time $t = 0$, these clocks are synchronized. The clock C moves from A to B and is read there. In the rest frame of A and B, C is slow.



Left: World lines of a light clock and a light beam in three different frames of reference.

Right: Three light clocks for measuring time dilation.

Invariant time or proper time of a space-time-point

The timing in the rest frame can be calculated by any observer using the Lorentz transformation. This proper time is invariant for every observer.

$$\text{LT into the rest frame:} \quad \begin{pmatrix} 0 \\ c\tau \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}, \quad \text{and } x = vt$$

$$\text{therefore:} \quad c\tau = \gamma\left(-\frac{xv}{c} + ct\right) = \gamma\left(-\frac{v^2 t}{c} + ct\right) = \gamma \underbrace{\left(-\frac{v^2}{c^2} + 1\right)}_{\gamma^{-2}} ct$$

$$\text{Proper time:} \quad \tau = \frac{1}{\gamma} t$$

(2-6)

Four-velocity

By differentiating the four-space in respect to the invariant time *t*, a four-vector with invariant length, the four-velocity, arises. Its length is $-c^2$, independent of the reference frame.

$$\text{Four-velocity:} \quad \vec{v}^{(4)} \equiv \frac{d\vec{x}^{(4)}}{d\tau} = \gamma \frac{d\vec{x}^{(4)}}{dt} = \gamma \frac{d}{dt}\left(\vec{x}, ct\right) = \gamma\left(\vec{v}, c\right)$$

$$\text{Length square (invariant):} \quad \vec{v}^{(4)2} = \gamma^2\left(g_{ij} v^i v^j + g_{44} c^2\right) = \underbrace{\gamma^2}_{\frac{1}{1-v^2/c^2} = \frac{c^2}{c^2-v^2}} \left(v^2 - c^2\right) = -c^2$$

(2-7)