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Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

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2.2 Lorentz transformation LT (Computational example)

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = LT\begin{pmatrix} x \\ ct \end{pmatrix}, \; LT = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

```
clear all
syms a_11 a_12 a_21 a_22 real
syms x ct v xb c ctb real
```

1 Basics

Four-space in reference system S: $\vec{r} = \begin{pmatrix} x \\ ct \end{pmatrix}$, in S': $\vec{r'} = \begin{pmatrix} x' \\ ct' \end{pmatrix}$

Metric: $G = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Lorentz transformation: $T_L = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

```
r_v=[x;ct]
```

r_v =

$$\begin{pmatrix} x \\ \mathrm{ct} \end{pmatrix}$$

Metric: $G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

```
G=[1 0;0 -1]
```

G = 2×2

```
1     0
0    -1
```

Approach for the Lorentz transformation $T_L = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

```
LT1=[a_11 a_12;a_21 a_22]
```

```
LT1 =
```

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

so

```
r_vb=LT1*r_v
```

```
r_vb =
```

$$\begin{pmatrix} a_{12}\,\mathrm{ct} + a_{11}\,x \\ a_{22}\,\mathrm{ct} + a_{21}\,x \end{pmatrix}$$

2 Transformation into the rest frame

Origin of S' in S: $\begin{pmatrix} 0 \\ ct' \end{pmatrix} = T_L \begin{pmatrix} x \\ ct \end{pmatrix} \quad \text{und} \quad x = vt$

so: $\vec{r}' = \begin{pmatrix} 0 \\ ct' \end{pmatrix}, \quad v = \frac{x}{t} = \frac{cx}{ct} \quad \rightarrow \quad a_{12}(a_{11})$

Rest frame

```
RS=[xb==0 x==ct*v/c]
```

```
RS =
```

$$\left(\mathrm{xb} = 0 \quad x = \frac{\mathrm{ct}\,v}{c} \right)$$

substitution in $\vec{r}\,'$

```
sube(r_vb,RS)
```

```
ans =
```

$$\begin{pmatrix} a_{12}\,\mathrm{ct} + \frac{a_{11}\,\mathrm{ct}\,v}{c} \\ a_{22}\,\mathrm{ct} + \frac{a_{21}\,\mathrm{ct}\,v}{c} \end{pmatrix}$$

```
a_12=solve(ans(1,1),a_12)
```

```
a_12 =
```

$$-\frac{a_{11}\,v}{c}$$

The Lorentz transformation is therefore

```
LT2=subs(LT1)
```

```
LT2 =
```

$$\begin{pmatrix} a_{11} & -\frac{a_{11}\,v}{c} \\ a_{21} & a_{22} \end{pmatrix}$$

and $\vec{r}\,'$

```
r_vb=eval(r_vb)
```

```
r_vb =
```

$$\begin{pmatrix} a_{11}\,x - \frac{a_{11}\,\mathrm{ct}\,v}{c} \\ a_{22}\,\mathrm{ct} + a_{21}\,x \end{pmatrix}$$

3 Lorentz transformation

Equating the lengths of the four-vectors leads to the other three equations:

Invariance: $\left(\vec{r}\right)^2 = \left(\vec{r}\,'\right)^2, \quad \left(\left(\vec{r}\right)^2 = g_{\mu\nu}x^\mu x^\nu = \vec{r}^{\,t}\,(g_{\mu\nu})\,\vec{r}\right)$

```
r_vb'*G*r_vb==r_v'*G*r_v
```

```
ans =
```

$$\left(a_{11}\,x - \frac{a_{11}\,\mathrm{ct}\,v}{c}\right)^2 - (a_{22}\,\mathrm{ct} + a_{21}\,x)^2 = x^2 - \mathrm{ct}^2$$

all to one side:

```
rhs(expand(ans-lhs(ans)))
```

```
ans =
```

$$x^2 - \mathrm{ct}^2 + {a_{22}}^2\,\mathrm{ct}^2 - {a_{11}}^2\,x^2 + {a_{21}}^2\,x^2 - \frac{{a_{11}}^2\,\mathrm{ct}^2\,v^2}{c^2} + 2\,a_{21}\,a_{22}\,\mathrm{ct}\,x + \frac{2\,{a_{11}}^2\,\mathrm{ct}\,v\,x}{c}$$

Coefficients of x^2, xct and $(ct)^2$

```
Coeff=coeffs(ans,[x ct])
```

```
Coeff =
```

$$\left({a_{22}}^2 - \frac{{a_{11}}^2\,v^2}{c^2} - 1 \quad 2\,a_{21}\,a_{22} + \frac{2\,{a_{11}}^2\,v}{c} \quad -{a_{11}}^2 + {a_{21}}^2 + 1 \right)$$

all coefficients have to be 0 $\rightarrow a_{ij}$

```
assumeAlso(v>0 & v<c)
Sol=solve(Coeff,[a_11 a_21 a_22],'ReturnConditions',true);
SN=1;
ME=[a_11==Sol.a_11(SN),a_21==Sol.a_21(SN),a_22==Sol.a_22(SN)];
LT=simplify(sube(LT2,ME),'Steps',20)
```

```
LT =
```

$$\begin{pmatrix} \frac{c}{\sqrt{c^2 - v^2}} & -\frac{v}{\sqrt{c^2 - v^2}} \\ -\frac{v}{\sqrt{c^2 - v^2}} & \frac{c}{\sqrt{c^2 - v^2}} \end{pmatrix}$$

Save LT for further use in Chap 2.3

```
save Kap02 LT
```