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Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

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3.1 Dynamics

For the formulation of classical mechanics, Isaac Newton distinguished sharply between content and dynamics. His dynamic laws apply in general, regardless of any content. They are then applied to the motion of bodies on trajectories, such as on the famous apple falling from the tree, the pendulum, or the planetary orbits.

The dynamic laws describe the connection between cause and effect and are based on the conservation of energy, mass and charge. In classical mechanics, these laws are applied to objects on trajectories, in quantum mechanics they apply to the response of detectors. In quantum mechanics, however, due to quantization, the phenomenon affects the dynamics and the strict Newtonian separation is abolished.

- The world appears in constant change (dynamics)
- Fundamental field:

Action S

↙ ↘

Dynamics classical	Dynamics quantum mechanical
(Action on objects)	(Action on measurement probability ρ)
$\vec{\nabla} S = \vec{p} = m\dot{\vec{r}}$	$\underbrace{\psi = \sqrt{\rho} e^{iS/\hbar}}_{\text{Wave function}}$

- Formulation of the dynamics:
  - Invariant
  - Uniform for classical mechanics and quantum mechanics

(3-1)

Four-momentum

The invariance of the speed of light leads to the Minkowski space with its four-vectors. The momentum as a basic dynamic quantity must be embedded in this space, this means it has to get a fourth component, such as four-space or four-velocity. This leads to the concept of energy.

$$\text{Four-momentum} \quad \vec{p}^{(4)} = \underbrace{p_\mu \vec{g}^{(4)\mu}}_{\vec{g}^{(4)i}=\hat{e}_i,\vec{g}^{(4)4}=-i\hat{e}_4} = \left(\vec{p}, -E/c\right)_{\vec{g}^{(4)}}$$

$$\text{Length of four-momentum (Invariant):} \quad \left(\vec{p}^{(4)}\right)^2 = g^{\mu\nu} p_\mu p_\nu = \vec{p}^2 - \frac{E^2}{c^2} = \underbrace{m_0^2}_{\substack{\text{Rest mass,}\\ \text{invariant mass}}} \underbrace{\left(\vec{v}^{(4)}\right)^2}_{-c^2} = \underbrace{-m_0^2 c^2}_{\substack{\text{const.}\\ \text{Lorentz-scalar}}}$$

$$\text{Hence:} \quad E = c\sqrt{m_0^2 c^2 + \vec{p}^2} = m_0 c^2 \sqrt{1 + \frac{\vec{p}^2}{m_0^2 c^2}}$$

$$\text{Small momenta:} \quad E = m_0 c^2 \left(1 + \frac{1}{2}\frac{\vec{p}^2}{m_0^2 c^2} + O\left(\frac{\vec{p}^4}{m_0^2 c^4}\right)\right) \simeq m_0 c^2 + \underbrace{\frac{\vec{p}^2}{2m_0}}_{\substack{\text{Kinetic energy}\\ \to E \text{ means energy}}}$$

$$\text{Einstein formula for } \vec{p} = 0: \quad E = m_0 c^2$$

(3-2)

We also see here how the equivalence of energy and mass, which is the basis of nuclear power, atomic or hydrogen bombs, follows alone from the invariance of the speed of light.

The Action field S

Momentum has direction, and is represented as a vector. The representation of the vector requires choosing a coordinate system with arbitrary base and coordinates. The scalar product of four-position and four-momentum is free from this arbitrariness. This leads to the concept of action.

The importance of the action *S* becomes apparent, for example, in quantum mechanics, where one considers quantas of action instead of particles. Plank's action quantum *h* also underlines the significance of this concept. Like the speed of light *c*, *h* is a fundamental constant in nature.

$$\text{Action field } S \text{ as scalar product of space and momentum (invariant):}$$

$$\text{Differential:} \quad dS \equiv \vec{p}^{(4)} \cdot d\vec{r}^{(4)} = \underbrace{\frac{\partial S}{\partial q^\mu}}_{\equiv p_\mu} dq^\mu = \underbrace{p_i \vec{g}^{(4)i} \cdot \vec{g}_j^{(4)} dq^j}_{p_j\, dq^j} - \underbrace{\frac{E}{c} \vec{g}^{(4)4} \cdot \vec{g}_4^{(4)} d\,ct}_{E\, dt}$$

$$\text{Hence:} \quad \begin{cases} \vec{\nabla} S = \vec{p}, \\ \dfrac{\partial S}{\partial t} = -E \end{cases}$$

$$\text{Force:} \quad F_i = \frac{\partial}{\partial t} \underbrace{p_i}_{\frac{\partial}{\partial q^i} S} = \frac{\partial}{\partial q^i} \underbrace{(-E)}_{\substack{\frac{\partial}{\partial t} S\\ \text{(Energy)}}}$$

(3-3)

Hamilton-Jacobi equation (HJE)

For the sake of simplicity, we supplement the energy here with a potential. This is later derived invariantly through the so-called minimal substitution. The energy now consists of kinetic and potential energy. Using the action field S, the law of conservation of energy can be expressed with the aid of the action field *S* in the form of the Hamilton-Jacobi equation.

$$HJE: \underbrace{\frac{\partial S(\vec{r},t)}{\partial t}}_{-E} + \underbrace{\frac{1}{2m}\left(\vec{\nabla} S(\vec{r},t)\right)^2}_{\frac{\vec{p}^2}{2m}} + \underbrace{V(\vec{r})}_{\substack{\text{Violates} \\ \text{spatial symmetry} \\ \rightarrow \text{change of momentum}}} = 0$$

(3-4)

Conservation of energy

Due to the potential *V*, the spatial symmetry is broken, but the temporal symmetry remains. The energy is a constant.

$$\begin{aligned} &\text{Approach:} && S(\vec{r},t) = S_{\vec{r}}(\vec{r}) + S_t(t) \\ &\text{Separation of the variables:} && \underbrace{\frac{1}{2m}\left(\vec{\nabla} S_{\vec{r}}(\vec{r})\right)^2 + V(\vec{r})}_{\text{time-independent}} = \underbrace{-\frac{\partial S_t(t)}{\partial t}}_{\text{location-independent}} = E \\ &\text{Hence:} && E = \text{constant} \end{aligned}$$

(3-5)

Free action (V=0)

$$\begin{aligned} &\text{Approach:} && S(\vec{r},t) = \underbrace{\vec{p}\cdot\vec{r}}_{S_r} - \underbrace{E\cdot t}_{-S_t} \\ &HJE: && \frac{\partial S}{\partial t} + \frac{\left(\vec{\nabla} S\right)^2}{2m} = -E + \frac{\vec{p}^2}{2m} = 0 \end{aligned}$$

(3-6)

Constants of motion

If the potential does not depend on one of the specific coordinates, then the canonical momenta of these coordinates are further conserved quantities. We will take a look at two examples.

Example: Oblique throw in homogeneous potential

Conservation of energy: $E=\frac{1}{2m}\left(\vec{\nabla}S\right)^2+V\left(y\right)$

Approach : $S\left(\vec{r},t\right)=S_x\left(x\right)+S_y\left(y\right)+S_t\left(t\right)$

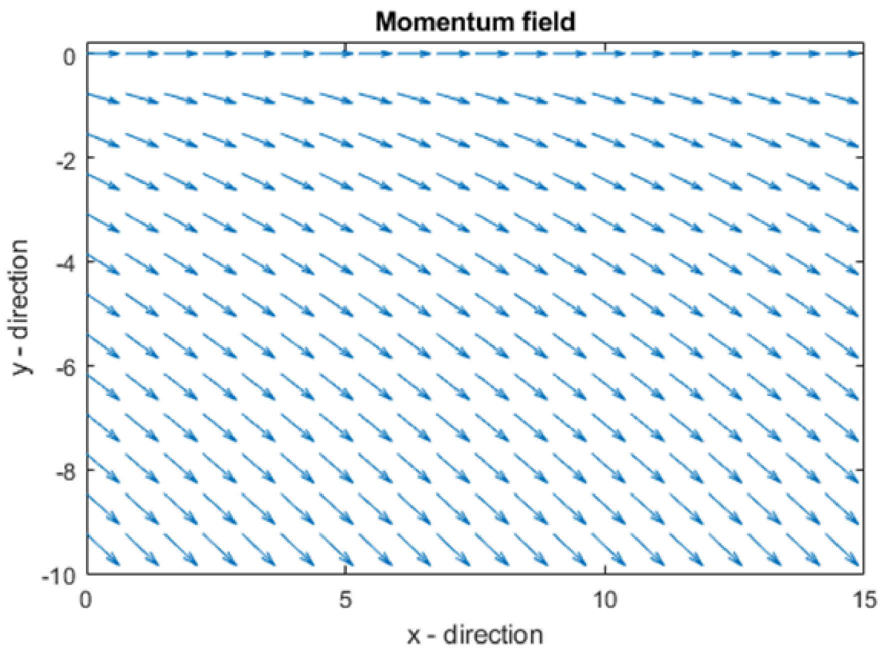
Hamilton-Jacobi equ.: $E=\frac{1}{2m}\left(\frac{\partial S_x\left(x\right)}{\partial x}\right)^2+\frac{1}{2m}\left(\frac{\partial S_y\left(y\right)}{\partial y}\right)^2+V\left(y\right)$

Seperation: $\underbrace{\left(\frac{\partial S_x\left(x\right)}{\partial x}\right)^2}_{F_x(x)}=\underbrace{2m\left(E-\frac{1}{2m}\left(\frac{\partial S_y\left(y\right)}{\partial y}\right)^2-V\left(y\right)\right)}_{F_r(r)}=const$

Constant of motion: $\frac{\partial S}{\partial x}=p_x=\pm\sqrt{2m\left(E-V\left(y\right)-\frac{p_y^2}{2m}\right)}=const$

Solve for $p_y\left(r\right)$: $p_y=\frac{\partial S_y\left(y\right)}{\partial y}=\pm\sqrt{2m\left(E-\frac{p_x^{\ 2}}{2m}-V\left(y\right)\right)}$

(3-7)



By separating the variables, the momentum can be derived as a function of space and energy and represented as a vector field by momentum arrows

Example: Central potential

Energy: $E = \frac{1}{2m}\left(\vec{\nabla} S\right)^2 + V\left(|\vec{r}|\right)$

Approach: $S(\vec{r},t) = S_r(r) + S_\varphi(\varphi) + S_t(t)$

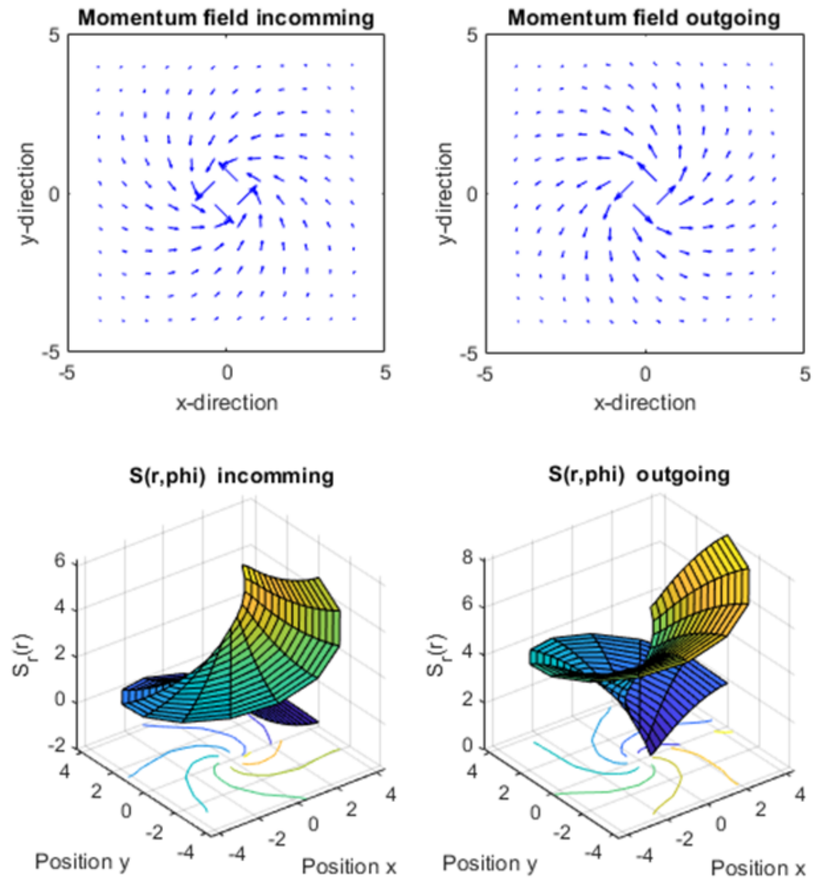
Hamilton-Jacobi-Eq.: $E = \frac{1}{2m}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S_\varphi(\varphi)}{\partial \varphi}\right)^2 + V(r)$

Separation: $\underbrace{\left(\frac{\partial S_\varphi(\varphi)}{\partial \varphi}\right)^2}_{F_\varphi(\varphi)} = \underbrace{2mr^2\left(E - \frac{1}{2m}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 - V(r)\right)}_{F_r(r)} = const$

Constant of motion: $\frac{\partial S}{\partial \varphi} = p_\varphi = \pm r\sqrt{2m(E-V) - \left(\frac{\partial S_r(r)}{\partial r}\right)^2} = const$

Solve for $p_r(r)$: $p_r = \frac{\partial S_r(r)}{\partial r} = \pm\sqrt{2m\left(E - V - \frac{{p_\varphi}^2}{2mr^2}\right)}$

(3-8)



Momentum field $\vec{p}$ and space part of the action field $S_{\vec{r}}$ for $V \propto \frac{1}{r}$ (Kepler/Coulomb)

Integrability

A dynamic system is called integrable if it can be made separable by the choice of suitable (generalized) coordinates, that is, decomposed into one-dimensional equations. These individual equations can be integrated to obtain unique solutions. In classical physics these are trajectories.

If the system is not separable, chaotic behavior is possible. In classical mechanics, closely adjacent trajectories can diverge exponentially. The specification of a unique trajectory is then no longer possible. This is already the case with a free moving ball in a stadium-like area (see Chap07_4).