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Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

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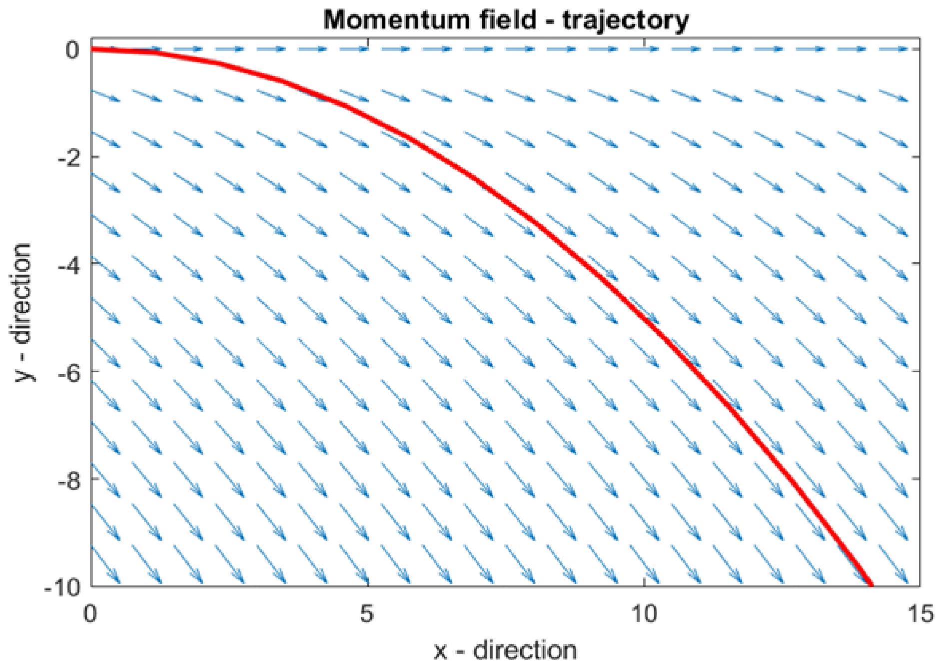
4.1 Classical mechanics

Trajectories

Classical mechanics deals with objects. The objects are characterized by a name or pointer, and by context-independent properties. Object properties are first the mass *m* and its position $\vec{r}$. The position changes continuously over time, resulting in the trajectory $\vec{r}(t)$ and velocity $\vec{v} = \dot{\vec{r}}$.

Content and dynamics are united by the requirement that for each point of the trajectory the velocity of the object is proportional to the momentum. Thus, the classical momentum becomes a function of time.

$$
\begin{array}{ll}
\text{Fundamentals of classical mechanics:} & \text{Objects (point particles)} \\
\text{Independent properties:} & \begin{array}{c|c|c} \text{Mass} & \text{Position} & \text{Velocity} \\ \hline m & \vec{r}(t) & \vec{v}(t) = \dot{\vec{r}}(t) \end{array} \\
\text{Unification of content and motion:} & \vec{p}(\vec{r}(t)) = \vec{p}(t) = m\dot{\vec{r}}(t) \\
\text{Trajectory:} & q^i(t) = q^i(t_0) + \frac{1}{m}\int_{t_0}^{t} \underbrace{g^{ik} p_k\left(q^j(t')\right)}_{mv^i(t')} dt'
\end{array}
\tag{4-1}
$$



A trajectory is laid over the momentum field so that the velocity vector is proportional to the momentum vector everywhere

In (4.1), the trajectory results from an integral equation which is generally difficult to solve. Usually, the trajectory is determined by a differential equation.

Newtonian equation of motion from energy conservation

$$\text{Energy:} \qquad E = \frac{\vec{p}^2}{2m} + V(\vec{r}) = const$$

$$\text{Time derivative:} \qquad 0 = \frac{dE}{dt} = \underbrace{\frac{\vec{p}}{m}}_{\dot{\vec{r}}}\dot{\vec{p}} + \vec{\nabla}V(\vec{r})\dot{\vec{r}} = \left(\dot{\vec{p}} + \vec{\nabla}V(\vec{r})\right)\dot{\vec{r}}$$

$$\text{Newton's equation of motion:} \qquad \dot{\vec{p}} = -\vec{\nabla}V(\vec{r}) = \vec{F}(\vec{r})$$

(4-2)

These equations can then be solved analytically or numerically.

Separation of the variables and integration of the momentum

In integrable systems, the separation of the individual degrees of freedom is achieved by selecting suitable coordinates. The trajectory can then be determined by solving one-dimensional integrals.

$$\text{Example:} \qquad E = \frac{p_i g^{ii} p_i}{2m} + \frac{p_j g^{jk} p_k}{2m} + V(q^i) \quad j,k \neq i$$

$$\text{Separated momentum:} \qquad p_i = \pm\sqrt{\frac{2m}{g^{ii}}\left(\underbrace{E - \frac{p_j g^{jk} p_k}{2m}}_{E_i = \text{const.}} - V(q^i)\right)}$$

(4-3)

$$\text{Time evolution:} \quad t - t_0 = \int_{t_0}^{t} dt' = \int_{t_0}^{t} \frac{\overbrace{m\dot{q}^i}^{1=}}{g^{ii} p_i} dt' = \pm \int_{q^i(t_0)}^{q^i(t)} \sqrt{\frac{m}{2g^{ii}\left(E_i - V\left(q^i\right)\right)}} dq^i$$

$$\text{Trajectory:} \quad q^i(t) \quad (\text{by inverting the time evolution})$$

(4-4)

Hamilton’s equations (HE)

This is the most general representation of classical physics. It also allows us to determine 'trajectories' for generalized coordinates and non-integrable systems. The Hamiltonian *H* is introduced as a field of generalized coordinates and momenta. If the function value of *H* is constantly equal to the energy, the space coordinates and momenta are implicitly linked. The additional demand for $\dot{q}^i = \frac{1}{m} g^{ij} p_j$ on the trajectory leads to the *HE*.

Hamiltonian	Momentum-position	further
$H(\vec{q}, \vec{p}) \equiv \frac{1}{2m} g^{ij} p_i p_j + V(\vec{q})$	$\underbrace{\dot{q}^i = \frac{1}{m} g^{ij} p_j}_{\vec{q}=\vec{q}(t),\ \ \vec{p}=\vec{p}(\vec{q}(t))=\vec{p}(t)}$	$\dot{q}^i = \frac{\partial H}{\partial p_i}$

(4-5)

On a trajectory, the Hamiltonian is equal to the energy, that is, constant.

Thereby it also follows the momentum change on the trajectory:

$$\dot{p}_i = \frac{d}{dt}\frac{\partial S}{\partial q^i} = \underbrace{\frac{\partial}{\partial q^i}\frac{\partial S}{\partial t} + \frac{\partial^2 S}{\partial q^i \partial q^j}\underbrace{\dot{q}^j}_{\frac{\partial H}{\partial p_j}}}_{S(q(t))\text{ on the trajectory!}} = \underbrace{\frac{\partial^2 S}{\partial q^i \partial t} + \underbrace{\frac{\partial^2 S}{\partial q^i \partial q^j}}_{\frac{\partial p_j}{\partial q^i}}\frac{\partial H}{\partial p_j} + \frac{\partial H}{\partial q^i}}_{\frac{d}{dq^i}\left(\frac{\partial S}{\partial t} + H\right) = \frac{d}{dq^i}(-E+H)=0} - \frac{\partial H}{\partial q^i} = -\frac{\partial H}{\partial q^i}$$

(4-6)

This is the popular formulation of classical mechanics:

$$HE: \begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q^i} \end{cases}$$

(4-7)

Cyclic variables

If the Hamiltonian does not depend on a coordinate q^i, then the conjugate momentum p_i is a conserved quantity, that is a constant of the motion.

$$
\begin{aligned}
&q^i \text{ cyclic:} && H \neq F\left(q^i\right) \\
&HE: && \dot{p}_i = -\frac{\partial H}{\partial q^i} = 0 \\
&\text{therefore:} && p_i = \text{const.}
\end{aligned}
$$

(4-8)

Generalization

All possible forms of classical systems can be handled by choosing suitable coordinates .

- Holonomic constraints (independent of time)
- Rheonomic constraints (time-dependent)
- Many-body systems
- Continuous mass distribution (hydrodynamic, structural mechanics)
- Statistical treatment of many-body systems (thermodynamics)

(4-9)