Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

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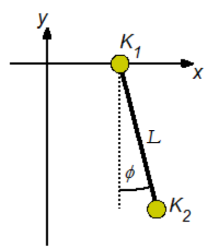
4.3 Roll pendulum (Computational example)

The suspension point K_1 of a plane pendulum slides frictionless along the x-axis. The pendulum body K_2 has the distance L from the suspension point. Both bodies have the same mass $m_1 = m_2 = m$ and the connection between K_1 and K_2 is massless



We choose as generalized coordinates x and ϕ.

Coordinates: $\vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ x + L\sin\varphi \\ l\cos\varphi \end{pmatrix}, \quad (y_1 = 0!)$

Parameters: mass, gravitational acceleration, pendulum length $m = g = L = 1$, energy $E = 0.9999$

```
clear all
syms x L m g phi E E0 positive
r_v=[x;x+L*sin(phi);-L*cos(phi)]
```

r_v =

$$\begin{pmatrix} x \\ x + L\sin(\varphi) \\ -L\cos(\varphi) \end{pmatrix}$$

```
Par=[E0==0.9999 m==1 g==1 L==1]
```

Par =

$$\left(E_0 = \frac{9999}{10000} \quad m = 1 \quad g = 1 \quad L = 1\right)$$

1 Ground vectors and metric

Jacobian *J*, metric *G* and contravariant metric G_i: $J = \left(\frac{\partial x_i}{\partial q_j}\right),\ G = J'J,\ G_i = G^{-1}$

```
J=jacobian(r_v,[x,phi])
```

J =

$$\begin{pmatrix} 1 & 0 \\ 1 & L\cos(\varphi) \\ 0 & L\sin(\varphi) \end{pmatrix}$$

```
G=simplify(J'*J)
```

G =

$$\begin{pmatrix} 2 & L\cos(\varphi) \\ L\cos(\varphi) & L^2 \end{pmatrix}$$

```
G_i=G^(-1)
```

G_i =

$$\begin{pmatrix} -\frac{1}{\cos(\varphi)^2 - 2} & -\frac{\cos(\varphi)}{2\,L - L\cos(\varphi)^2} \\ -\frac{\cos(\varphi)}{2\,L - L\cos(\varphi)^2} & -\frac{2}{L^2\cos(\varphi)^2 - 2\,L^2} \end{pmatrix}$$

2 Hamiltonian

$H = \frac{1}{2m}(p_r, p_\varphi)G_i\begin{pmatrix} p_r \\ p_\varphi \end{pmatrix} + V(r)$ mit $V(r) = -mgL\cos\varphi$

```
syms p_x p_phi real
p_v=[p_x;p_phi]
```

p_v =

$$\begin{pmatrix} p_x \\ p_\varphi \end{pmatrix}$$

```
H=1/(2*m)*(p_v'*G_i*p_v)-m*g*L*cos(phi);
H=expand(H)
```

H =

$$\frac{{p_\varphi}^2}{2\,L^2\,m-L^2\,m\cos(\varphi)^2}+\frac{{p_x}^2}{2\,(2\,m-m\cos(\varphi)^2)}-\frac{p_\varphi\,p_x\cos(\varphi)}{2\,L\,m-L\,m\cos(\varphi)^2}-L\,g\,m\cos(\varphi)$$

p_x is a constant of motion: $\frac{\partial p_x}{\partial t}=-\frac{\partial H}{\partial x}=0$

```
-diff(H,x)
```

ans = 0

3 Special case $\vec{r}=\begin{pmatrix}x_1\\x_2\\y_2\end{pmatrix}=\begin{pmatrix}x\\x+L\sin\varphi\\l\cos\varphi\end{pmatrix},\quad (y_1=0!)\;p_x=0$

We calculate for this case the angular momentum p_ϕ as a function of ϕ and the energy $H=E$ and from this the reversal points $\phi_0=\phi|_{p_\phi=0}$.

Hamiltonian

```
H3=sube(H,p_x==0)
```

H3 =

$$\frac{{p_\varphi}^2}{2\,L^2\,m-L^2\,m\cos(\varphi)^2}-L\,g\,m\cos(\varphi)$$

Momentum p_ϕ from $E=H(\phi,p_\phi)\;\rightarrow\;p_\phi(\phi,E)$ for movement from left to right

```
p3_phi=solve(E0==H3,p_phi,'IgnoreProperties',true)
```

p3_phi =

$$\begin{pmatrix}L\,\sqrt{-m}\,\sqrt{\cos(\varphi)^2-2}\,\sqrt{E_0+L\,g\,m\cos(\varphi)}\\-L\,\sqrt{-m}\,\sqrt{\cos(\varphi)^2-2}\,\sqrt{E_0+L\,g\,m\cos(\varphi)}\end{pmatrix}$$

```
double(sube(p3_phi,[Par,phi==0])) %p3_phi(2) from left to right
```

```
ans = 2×1
   -1.4142
    1.4142
```

```
p3_phi=p3_phi(2)
```

p3_phi = $-L\,\sqrt{-m}\,\sqrt{\cos(\varphi)^2-2}\,\sqrt{E_0+L\,g\,m\cos(\varphi)}$

Reversal points ϕ_0 at $p_\phi(\phi, E) = 0$

```
phi0=solve(p3_phi,phi,'IgnoreProperties', true)
```

```
phi0 =
```

$$\begin{pmatrix} \pi + \operatorname{acos}\left(\frac{E_0}{L\,g\,m}\right) \\ \pi - \operatorname{acos}\left(\frac{E_0}{L\,g\,m}\right) \end{pmatrix}$$

```
double(sube(phi0,Par))
```

```
ans = 2×1
    3.1557
    3.1275
```

```
phi0=min(ans)
```

```
phi0 = 3.1275
```

4 Period time

$$T = 2\int_{t(-|\varphi_0|)}^{t(|\varphi_0|)} dt = 2\int_{t(-|\varphi_0|)}^{t(|\varphi_0|)} \frac{\dot{\varphi}}{\dot{\varphi}}dt = 2\int_{-|\varphi_0|}^{|\varphi_0|} \frac{1}{\dot{\varphi}}d\varphi$$

```
dphi_t=diff(H3,p_phi)
```

```
dphi_t =
```

$$\frac{2\,p_\varphi}{2\,L^2\,m - L^2\,m\cos(\varphi)^2}$$

```
sube(dphi_t,[p_phi==p3_phi,Par]);
T=double(vpaintegral(2/ans,phi,-phi0,phi0))
```

```
T = 19.5765
```

5 Equations of motion

Equations: $\dot{x} = \frac{\partial H}{\partial p_x},\ \dot{\varphi} = \frac{\partial H}{\partial p_\varphi},\ \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi}$

Initial values: $x(0) = 0,\ p_\varphi(0) = 0,\ \varphi(0) = \varphi_0$

```
dx_t=subs(diff(H,p_x),p_x,0)
```

```
dx_t =
```

$$-\frac{p_\varphi \cos(\varphi)}{2\,L\,m - L\,m\cos(\varphi)^2}$$

```
dphi_t=subs(diff(H,p_phi),p_x,0)
```

```
dphi_t =
```

$$\frac{2\,p_\varphi}{2\,L^2\,m - L^2\,m\cos(\varphi)^2}$$

```
dp_phi_t=-subs(diff(H,phi),p_x,0)
```

```
dp_phi_t =
```

$$\frac{2\,L^2\,m\,{p_\varphi}^2\cos(\varphi)\sin(\varphi)}{\left(2\,L^2\,m - L^2\,m\cos(\varphi)^2\right)^2} - L\,g\,m\sin(\varphi)$$

6 Numerical solution

```
syms x_n(t) phi_n(t) p_phi_n(t)
var=[x==x_n(t) phi==phi_n(t) p_phi==p_phi_n(t)]
```

var = $\begin{pmatrix} x = x_n(t) & \varphi = \varphi_n(t) & p_\varphi = p_{\varphi,n}(t) \end{pmatrix}$

```
sube([dx_t;dphi_t;dp_phi_t],[var,Par])
```

```
ans =
```

$$\begin{pmatrix} \dfrac{\cos(\varphi_n(t))\,p_{\varphi,n}(t)}{\cos(\varphi_n(t))^2 - 2} \\ -\dfrac{2\,p_{\varphi,n}(t)}{\cos(\varphi_n(t))^2 - 2} \\ \dfrac{2\cos(\varphi_n(t))\sin(\varphi_n(t))\,p_{\varphi,n}(t)^2}{\left(\cos(\varphi_n(t))^2 - 2\right)^2} - \sin(\varphi_n(t)) \end{pmatrix}$$

```
f=odeFunction(ans,[x_n(t) phi_n(t) p_phi_n(t)])
```

```
f = function_handle with value:
    @(t,in2)[(in2(3,:).*cos(in2(2,:)))./(cos(in2(2,:)).^2-2.0);(in2(3,:).*-2.0)./(cos(in2(2,:)).^2-2.0);-sin(in2(2,
```

```
[t sol]=ode23s(f,linspace(0,T,100),[0;-phi0;0],odeset('RelTol',1e-8));
X=sol(:,1);Phi=sol(:,2);P_phi=sol(:,3);
```

7 Momentum

$p(t) = \sqrt{(p_x, p_\phi) G_i \begin{pmatrix} p_x \\ p_\phi \end{pmatrix}}$ für $p_x = 0$

```
p_v
```

```
p_v =
```

$$\begin{pmatrix} p_x \\ p_\varphi \end{pmatrix}$$

```
p7=sqrt(p_v'*G_i*p_v)
```

```
p7 =
```

$$\sqrt{-p_x \left(\frac{p_x}{\cos(\varphi)^2 - 2} + \frac{p_\varphi \cos(\varphi)}{2\,L - L\cos(\varphi)^2}\right) - p_\varphi \left(\frac{2\,p_\varphi}{L^2\cos(\varphi)^2 - 2\,L^2} + \frac{p_x \cos(\varphi)}{2\,L - L\cos(\varphi)^2}\right)}$$

```
sube(p7,[p_x==0 Par])
```

```
ans =
```

$$\sqrt{-\frac{2\,{p_\varphi}^2}{\cos(\varphi)^2 - 2}}$$

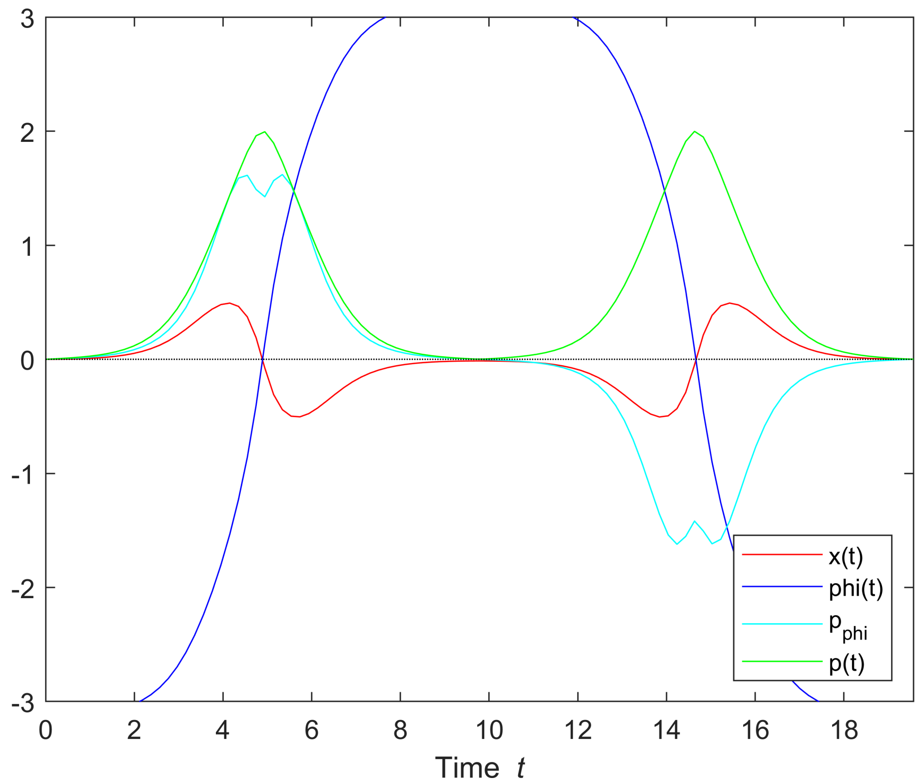
```
matlabFunction(ans)
```

```
ans = function_handle with value:
    @(p_phi,phi)sqrt((p_phi.^2.*-2.0)./(cos(phi).^2-2.0))
```

```
p=ans(P_phi,Phi);
```

8 Graphics

```
plot(t,X,'r',t,Phi,'b',t,P_phi,'c',t,p,'g',[0 T],[0 0],':k')
axis([0 double(T) 3*[- 1 1]])
xlabel('Time \sl t')
legend('x(t)','phi(t)','p_{phi}','p(t)','Location','southeast')
```



9 Animation → 'Roll_Pendulum.gif'

```
r_v
```

r_v =

$$\begin{pmatrix} x \\ x + L\sin(\varphi) \\ -L\cos(\varphi) \end{pmatrix}$$

```
r_vp=sube(r_v,Par)
```

r_vp =

$$\begin{pmatrix} x \\ x + \sin(\varphi) \\ -\cos(\varphi) \end{pmatrix}$$

```
matlabFunction(r_vp(1)),x1=ans(X);
```

```
ans = function_handle with value:
    @(x)x
```

```
matlabFunction(r_vp(2)),x2=ans(Phi,X);
```

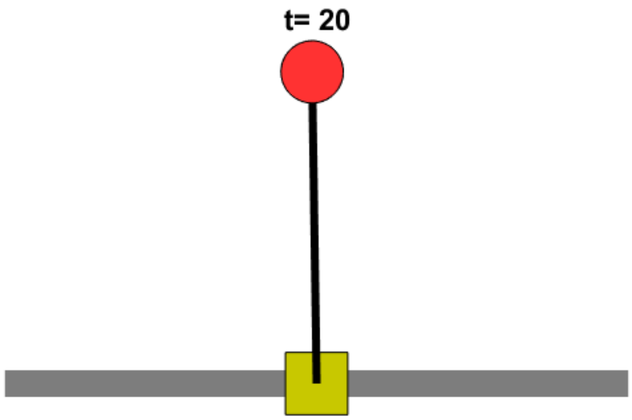
```
ans = function_handle with value:
    @(phi,x)x+sin(phi)
```

```
matlabFunction(r_vp(3)),y2=ans(Phi);
```

```
ans = function_handle with value:
    @(phi)-cos(phi)
```

```
c1=.8*[1 1 0]; c2=[1 .2 .2];
r=0.1; n=1;
figure
hold on
plot([-1 1],[0 0],'Color',.5*[1 1 1],'LineWidth',10)
H1=rectangle('Position',r*[-1 -1 2 2]+[x1(n) 0 0 0],'FaceColor',c1);
H2=plot([x1(n) x2(n)],[0 y2(n)],'k','LineWidth',3);
H3=rectangle('Position',r*[-1 -1 2 2]+[x2(n) y2(n) 0 0],'Curvature',[1 1],'FaceColor',c2);
Ht=title(sprintf('t= %2.1f',t(n)));
axis([-1.1 1.1 -1.1 1.1]); axis square; axis off, drawnow
im{1}=frame2im(getframe);

for n=2:size(x1,1)
    H1.Position=r*[-1 -1 2 2]+[x1(n) 0 0 0];
    H2.XData=[x1(n) x2(n)]; H2.YData=[0 y2(n)];
    H3.Position=r*[-1 -1 2 2]+[x2(n) y2(n) 0 0];
    Ht.String=sprintf('t= %2.0f',t(n));
    im{n} = frame2im(getframe);
end
```




```
FiNa='Roll_Pendulum.gif';
n=1;
[A,map] = rgb2ind(im{n},8);
imwrite(A,map,FiNa,'gif','LoopCount',Inf,'DelayTime',0);
for n = 2:size(im,2)
    [A,map] = rgb2ind(im{n},8);
    imwrite(A,map,FiNa,'gif','WriteMode','append','DelayTime',0);
end
```