EBERHARD KARLS UNIVERSITÄT TÜBINGEN

MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT
Fachbereich Physik

Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

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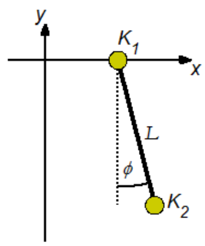
4.3 Roll pendulum (Computational example)

The suspension point K_1 of a plane pendulum slides frictionless along the x-axis. The pendulum body K_2 has the distance *L* from the suspension point. Both bodies have the same mass $m_1 = m_2 = m$ and the connection between K_1 and K_2 is massless



We choose as generalized coordinates *x* and *ϕ*.

Coordinates: $\vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ x + L\sin\varphi \\ l\cos\varphi \end{pmatrix}, \quad (y_1 = 0!)$

Parameters: mass, gravitational acceleration, pendulum length $m = g = L = 1$, energy $E = 0.9999$

```
clear all
syms x L m g phi E E0 positive
r_v=[x;x+L*sin(phi);-L*cos(phi)]
```

r_v =

$$\begin{pmatrix} x \\ x + L\sin(\phi) \\ -L\cos(\phi) \end{pmatrix}$$

```
Par=[E0==0.9999 m==1 g==1 L==1]
```

Par =

$$\left(E_0 = \frac{9999}{10000} \quad m = 1 \quad g = 1 \quad L = 1\right)$$

1 Ground vectors and metric

Jacobian *J*, metric *G* and contravariant metric G_i: $J = \left(\frac{\partial x_i}{\partial q_j}\right)$, $G = J'J$, $G_i = G^{-1}$

```
J=jacobian(r_v,[x,phi])
```

J =

$$\begin{pmatrix} 1 & 0 \\ 1 & L\cos(\phi) \\ 0 & L\sin(\phi) \end{pmatrix}$$

```
G=simplify(J'*J)
```

G =

$$\begin{pmatrix} 2 & L\cos(\phi) \\ L\cos(\phi) & L^2 \end{pmatrix}$$

```
G_i=G^(-1)
```

G_i =

$$\begin{pmatrix} -\frac{1}{\cos(\phi)^2 - 2} & -\frac{\cos(\phi)}{2\,L - L\cos(\phi)^2} \\ -\frac{\cos(\phi)}{2\,L - L\cos(\phi)^2} & -\frac{2}{L^2\cos(\phi)^2 - 2\,L^2} \end{pmatrix}$$

2 Hamiltonian

$$H = \frac{1}{2m}(p_r, p_\varphi)G_i\begin{pmatrix} p_r \\ p_\varphi \end{pmatrix} + V(r) \text{ mit } V(r) = -mgL\cos\varphi$$

```
syms p_x p_phi real
p_v=[p_x;p_phi]
```

p_v =

$$\begin{pmatrix} p_x \\ p_\phi \end{pmatrix}$$

```
H=1/(2*m)*(p_v'*G_i*p_v)-m*g*L*cos(phi);
H=expand(H)
```

H =

$$\frac{{p_\phi}^2}{2\,L^2\,m - L^2\,m\cos(\phi)^2} + \frac{{p_x}^2}{2\,(2\,m - m\cos(\phi)^2)} - \frac{p_\phi\,p_x\cos(\phi)}{2\,L\,m - L\,m\cos(\phi)^2} - L\,g\,m\cos(\phi)$$

p_x is a constant of motion: $\frac{\partial p_x}{\partial t} = -\frac{\partial H}{\partial x} = 0$

```
-diff(H,x)
```

```
ans = 
```
0

3 Special case $\vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ x + L\sin\varphi \\ l\cos\varphi \end{pmatrix}, \quad (y_1 = 0!)\; p_x = 0$

We calculate for this case the angular momentum p_ϕ as a function of ϕ and the energy $H = E$ and from this the reversal points $\phi_0 = \phi|_{p_\phi=0}$.

Hamiltonian

```
H3=sube(H,p_x==0)
```

```
H3 =
```
$$\frac{{p_\phi}^2}{2\,L^2\,m - L^2\,m\cos(\phi)^2} - L\,g\,m\cos(\phi)$$

Momentum p_ϕ from $E = H(\phi, p_\phi) \;\; \rightarrow \; p_\phi(\phi, E)$ for movement from left to right

```
p3_phi=solve(E0==H3,p_phi,'IgnoreProperties',true)
```

```
p3_phi =
```
$$\begin{pmatrix} L\,\sqrt{-m}\,\sqrt{\cos(\phi)^2 - 2}\,\sqrt{E_0 + L\,g\,m\cos(\phi)} \\ -L\,\sqrt{-m}\,\sqrt{\cos(\phi)^2 - 2}\,\sqrt{E_0 + L\,g\,m\cos(\phi)} \end{pmatrix}$$

```
double(sube(p3_phi,[Par,phi==0])) %p3_phi(2) from left to right
```

```
ans = 2×1
   -1.4142
    1.4142
```

```
p3_phi=p3_phi(2)
```

```
p3_phi = 
```
$-L\,\sqrt{-m}\,\sqrt{\cos(\phi)^2 - 2}\,\sqrt{E_0 + L\,g\,m\cos(\phi)}$

Reversal points ϕ_0 at $p_\phi(\phi, E) = 0$

```
phi0=solve(p3_phi,phi,'IgnoreProperties', true)
```

```
phi0 =
```

$$\begin{pmatrix} \pi + \mathrm{acos}\left(\frac{E_0}{L\,g\,m}\right) \\ \pi - \mathrm{acos}\left(\frac{E_0}{L\,g\,m}\right) \end{pmatrix}$$

```
double(sube(phi0,Par))
```

```
ans = 2×1
    3.1557
    3.1275
```

```
phi0=min(ans)
```

```
phi0 = 3.1275
```

4 Period time

$$T = 2\int_{t(-|\varphi_0|)}^{t(|\varphi_0|)} dt = 2\int_{t(-|\varphi_0|)}^{t(|\varphi_0|)} \frac{\dot{\varphi}}{\dot{\varphi}}dt = 2\int_{-|\varphi_0|}^{|\varphi_0|} \frac{1}{\dot{\varphi}}d\varphi$$

```
dphi_t=diff(H3,p_phi)
```

dphi_t =

$$\frac{2\,p_\phi}{2\,L^2\,m - L^2\,m\cos(\phi)^2}$$

```
sube(dphi_t,[p_phi==p3_phi,Par]);
T=double(vpaintegral(2/ans,phi,-phi0,phi0))
```

```
T = 19.5765
```

5 Equations of motion

Equations: $\dot{x} = \frac{\partial H}{\partial p_x},\ \dot{\varphi} = \frac{\partial H}{\partial p_\varphi},\ \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi}$

Initial values: $x(0) = 0,\ p_\varphi(0) = 0,\ \varphi(0) = \varphi_0$

```
dx_t=subs(diff(H,p_x),p_x,0)
```

dx_t =

$$-\frac{p_\phi\cos(\phi)}{2\,L\,m - L\,m\cos(\phi)^2}$$

```
dphi_t=subs(diff(H,p_phi),p_x,0)
```

dphi_t =

$$\frac{2\,p_\phi}{2\,L^2\,m - L^2\,m\cos(\phi)^2}$$

```
dp_phi_t=-subs(diff(H,phi),p_x,0)
```

dp_phi_t =

$$\frac{2\,L^2\,m\,{p_\phi}^2\cos(\phi)\,\sin(\phi)}{{(2\,L^2\,m-L^2\,m\cos(\phi)^2)}^2}-L\,g\,m\sin(\phi)$$

6 Numerical solution

```
syms x_n(t) phi_n(t) p_phi_n(t)
var=[x==x_n(t) phi==phi_n(t) p_phi==p_phi_n(t)]
```

var = $\left(x=x_n(t) \quad \phi=\phi_n(t) \quad p_\phi=p_{\phi,n}(t)\right)$

```
sube([dx_t;dphi_t;dp_phi_t],[var,Par])
```

ans =

$$\left(\begin{array}{c} \frac{\cos(\phi_n(t))\,p_{\phi,n}(t)}{\cos(\phi_n(t))^2-2} \\ -\frac{2\,p_{\phi,n}(t)}{\cos(\phi_n(t))^2-2} \\ \frac{2\cos(\phi_n(t))\,\sin(\phi_n(t))\,p_{\phi,n}(t)^2}{{\left(\cos(\phi_n(t))^2-2\right)}^2}-\sin(\phi_n(t)) \end{array}\right)$$

```
f=odeFunction(ans,[x_n(t) phi_n(t) p_phi_n(t)])
```

```
f = function_handle with value:
    @(t,in2)[(in2(3,:).*cos(in2(2,:)))./(cos(in2(2,:)).^2-2.0);(in2(3,:).*-2.0)./(cos(in2(2,:)).^2-2.0);-sin(in2(2,:
```

```
[t sol]=ode23s(f,linspace(0,T,100),[0;-phi0;0],odeset('RelTol',1e-8));
X=sol(:,1);Phi=sol(:,2);P_phi=sol(:,3);
```

7 Momentum

$p(t)=\sqrt{(p_x,p_\phi)G_i\begin{pmatrix}p_x\\p_\phi\end{pmatrix}}$ für $p_x=0$

```
p_v
```

p_v =

$$\begin{pmatrix}p_x\\p_\phi\end{pmatrix}$$

```
p7=sqrt(p_v'*G_i*p_v)
```

p7 =

$$\sqrt{-p_x\left(\frac{p_x}{\cos(\phi)^2-2}+\frac{p_\phi\cos(\phi)}{2\,L-L\cos(\phi)^2}\right)-p_\phi\left(\frac{2\,p_\phi}{L^2\cos(\phi)^2-2\,L^2}+\frac{p_x\cos(\phi)}{2\,L-L\cos(\phi)^2}\right)}$$

```
sube(p7,[p_x==0 Par])
```

ans =

$$\sqrt{-\frac{2\,{p_\phi}^2}{\cos(\phi)^2-2}}$$

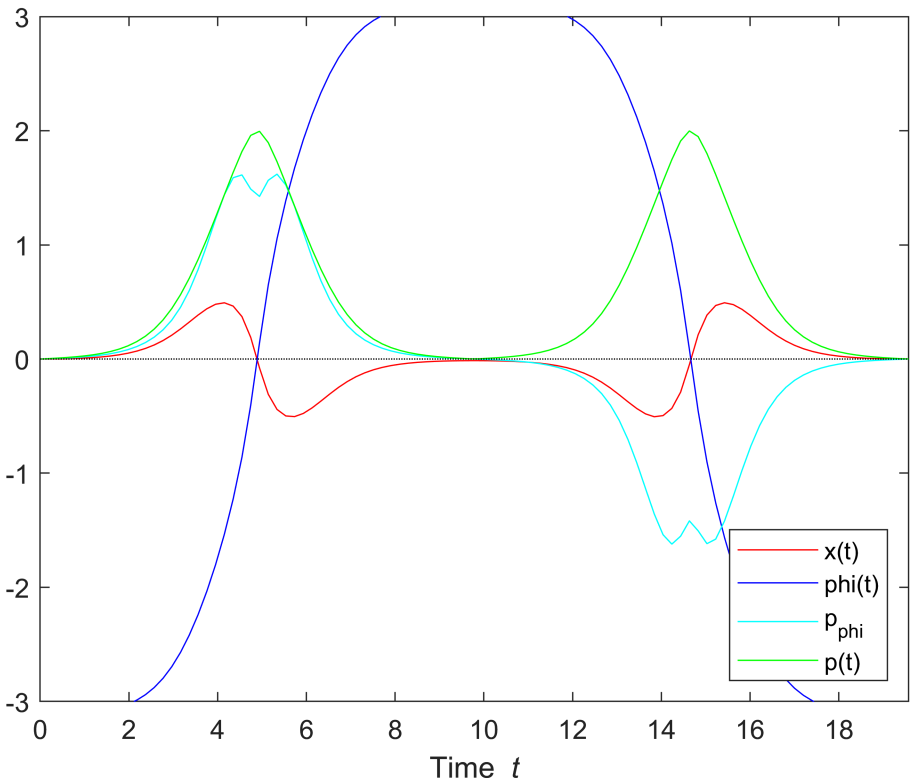
```
matlabFunction(ans)
```

```
ans = function_handle with value:
    @(p_phi,phi)sqrt((p_phi.^2.*-2.0)./(cos(phi).^2-2.0))
```

```
p=ans(P_phi,Phi);
```

8 Graphics

```
plot(t,X,'r',t,Phi,'b',t,P_phi,'c',t,p,'g',[0 T],[0 0],':k')
axis([0 double(T) 3*[- 1 1]])
xlabel('Time \sl t')
legend('x(t)','phi(t)','p_{phi}','p(t)','Location','southeast')
hold off
```



9 Animation → 'Roll_Pendulum.gif'

```
r_v
```

```
r_v =
```

$$\begin{pmatrix} x \\ x + L\sin(\phi) \\ -L\cos(\phi) \end{pmatrix}$$

```
r_vp=sube(r_v,Par)
```

r_vp =

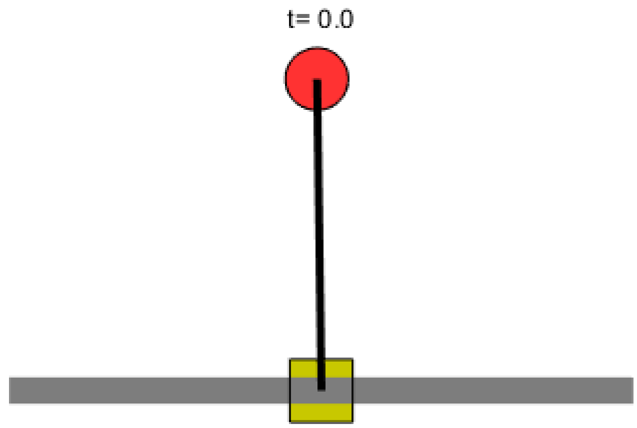
$$\begin{pmatrix} x \\ x + \sin(\phi) \\ -\cos(\phi) \end{pmatrix}$$

```
Nt=size(X,1);
```

```
s=matlabFunction(r_vp(1));x1=s(X);fx1=@(Ta)x1(round(1+(Nt-1)*Ta));
s=matlabFunction(r_vp(2));x2=s(Phi,X);fx2=@(Ta)x2(round(1+(Nt-1)*Ta));
s=matlabFunction(r_vp(3));y2=s(Phi);fy2=@(Ta)y2(round(1+(Nt-1)*Ta));
ft=@(Ta)t(round(1+(Nt-1)*Ta));
```

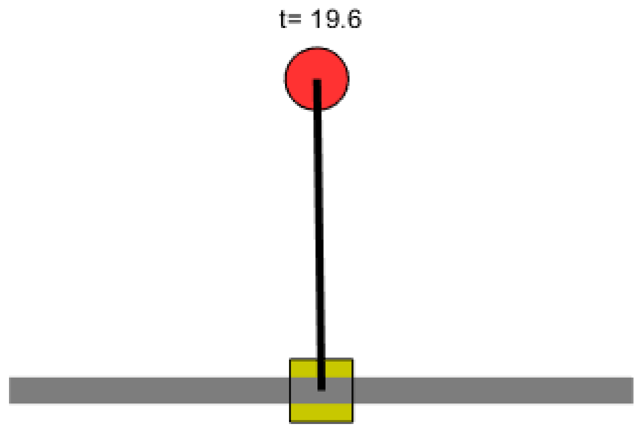
```
c1=.8*[1 1 0]; c2=[1 .2 .2];
r=0.1; n=1;

figure
hold on
plot([-1 1],[0 0],'Color',.5*[1 1 1],'LineWidth',10)
fanimator( @(Ta)plot([fx1(Ta) fx2(Ta)],[0 fy2(Ta)],'k','LineWidth',3), 'AnimationRange',[0 1],'
fanimator( @(Ta)rectangle('Position',r*[-1 -1 2 2]+[fx1(Ta) 0 0 0],'FaceColor',c1), 'AnimationR
fanimator( @(Ta)rectangle('Position',r*[-1 -1 2 2]+[fx2(Ta) fy2(Ta) 0 0],'Curvature',[1 1],'Fac
fanimator( @(Ta)text(0,1.2,sprintf('t= %2.1f',ft(Ta)),'HorizontalAlignment','center'), 'Animati
axis([-1.1 1.1 -1.1 1.1]); axis square; axis off
hold off
```




```
% playAnimation
```

```
writeAnimation('Roll_Pendulum.gif','Loopcount',inf)
```


t= 19.6