EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

MATHEMATISCH-
NATURWISSENSCHAFTLICHE FAKULTÄT
Fachbereich Physik

Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

Kurt Bräuer

6.1 Interaction

There are several ways to describe interaction.

Special relativity:	$E \simeq \frac{p^2}{2m} + m_0 c^2$	Free motion
Hamilton-Jacobi equation :	$\frac{\partial S}{\partial t} + \frac{\left(\vec{\nabla} S\right)^2}{2m} + V = 0$	Potential V
Electrodynamics:	$\underbrace{\frac{\partial}{\partial q_\mu} \underbrace{\frac{\partial}{\partial q^\mu} p_\nu}_{\text{Momentum gradient}}}_{\text{Continuity of the momentum gradient}} = \Box p_\nu = j \rightarrow \underbrace{\Box A_\nu = j_\nu}_{\text{Potential equation of electrodynamics}}$	Continuity of the momentum gradient
Curvature of space:	$\underbrace{e \underbrace{\frac{\partial A_\nu}{\partial q^\mu}}_{=\frac{\partial p_\nu}{\partial q^\mu}} = -m_0 g_{\mu\lambda} \Gamma^\lambda_{\kappa\nu} v^\kappa}_{\text{Principle of equivalence}}, \quad \underbrace{\Gamma^\lambda_{\mu\nu} = \vec{g}^{(4)\lambda} \cdot \frac{\partial \vec{g}^{(4)}_\mu}{\partial q^\nu}}_{\text{Christoffel symbols}}$	Equation for the metric

(6-1)

The connection of the interaction with the momentum gradient in electrodynamics and space curvature will be considered in more detail below.

In our conscious experience of the world, we constantly observe a change of momentum. This change is described physically by a momentum gradient. This gradient must be subject to a clear law, otherwise the change is scientifically or consciously intangible. All basic laws of electrodynamics and gravitation can be explained by the continuity of the momentum gradients.

Comparison of the interaction with the water outflow

Water flows out of a tap and spreads evenly on a surface, the electric field flows from a source and spreads out spatially:

Continuity equation	Integration	Therefore
$\underbrace{\underbrace{\dot{\rho}}_{\substack{=0 \\ \text{(Incompressible)}}} + \vec{\nabla}\cdot\rho\vec{v} = q\delta(\vec{r})}_{\text{Water}}$	$\underbrace{\int_{\text{Circular area}} \vec{\nabla}\cdot\rho\vec{v}d\vec{f}}_{=\int_{Circle}\rho\vec{v}\cdot\hat{e}_r ds=\rho v 2\pi R} = q \underbrace{\int_{\text{Circular area}} \delta(\vec{r})d^2\vec{r}}_{=1}$	$\rho v = \frac{q}{2\pi R}$
$\underbrace{\underbrace{\dot{E}}_{\substack{=0 \\ \text{(Stationary)}}} + \vec{\nabla}\cdot E\vec{c} = q\delta(\vec{r})}_{\text{Change of momentum (E-Field)}}$	$\underbrace{\int_{Sphere} \vec{\nabla}\cdot E\vec{c}d^3\vec{r}}_{=\int_{Surface} Ecd^2 f=Ec4\pi R^2} = q \underbrace{\int_{Sphere} \delta(\vec{r})d^2\vec{r}}_{=1}$	$\underbrace{E=\frac{1}{c}\frac{q}{4\pi R^2}}_{\substack{\text{Coulomb law} \\ \text{Gravitational law (Newton)}}}$

(6-2)

The water flow decreases radially in two dimensions with *1/R*, and the electric field in three dimensions with $1/R^2$. This results in each case from the continuity equation.

Invariant formulation of the continuity equation

We first write the continuity equation with four-vectors:

$$\text{Continuity equation:} \quad 0=\frac{\partial}{\partial t}\rho+\vec{\nabla}\cdot\rho\vec{v}=\underbrace{\left(\vec{\nabla},\frac{\partial}{\partial ct}\right)}_{\substack{\equiv\vec{\nabla}^{(4)}=\vec{g}^{\mu}\frac{\partial}{\partial q^{\mu}} \\ \text{(contravariant)}}}\cdot\underbrace{\left(\rho\vec{v},\rho c\right)}_{\substack{\equiv\vec{j}^{(4)} \\ \text{(covariant)}}}$$

$$=\frac{\partial}{\partial q^{\mu}}j^{\mu} \quad \text{(invariant under the Lorentz transformation}$$

$$\text{Four-current:} \quad j^{(4)}\equiv\rho\left(\vec{v},c\right)=\underbrace{\rho\gamma^{-1}}_{\equiv\rho_0}\underbrace{\overbrace{\gamma}^{\frac{1}{\sqrt{1-v^2/c^2}}}\left(\vec{v},c\right)}_{(v)^{(4)}}=\rho_0\vec{v}^{(4)}$$

$$\text{Invariant charge:} \quad \rho dV=\underbrace{\rho_0}_{\rho\gamma^{-1}}\underbrace{dV_0}_{\gamma dV}$$

(6-3)

The potential equations for electrodynamics and gravitation

The continuity equations for the components of the four-momentum gradient correspond to the potential equations of electrodynamics.

$$\text{Four-momentum } (p_\mu): \quad dS = \frac{\partial S}{\partial q^\mu} dq^\mu = p_\mu dq^\mu \quad \left(p^\mu p_\mu = -m_0 c^2\right)$$

$$\text{Observation: Momentum is generally not constant!} \quad \underbrace{\frac{\partial}{\partial q^\mu} p_\nu}_{\text{Momentum gradient}} \neq 0$$

$$\text{No arbitrariness} \rightarrow \text{Continuity equation:} \quad \underbrace{\frac{\partial}{\partial q_\mu}\frac{\partial}{\partial q^\mu}}_{\substack{\equiv\Box \\ \left(\text{D'Alembert-Operator}\right)}} p_\nu = \Box p_\nu \propto \underbrace{j_\nu}_{\text{Source current}}$$

(6-4)

However, in electrodynamics the vector potential A_μ is introduced and linked to the momentum through what is commonly known as 'minimal substitution'. The conserved total momentum is now $p_\mu - A_\mu$.

$$\text{Breaking of symmetry:} \quad \frac{\partial S}{\partial q^\mu} \rightarrow \underbrace{p_\mu - \underset{\text{Charge}}{e} A_\mu}_{\substack{\text{Conserved Lorentz scalar is now} \\ \left(\vec{\nabla}^{(4)} S\right)^2 = \left(\vec{p}^{(4)} - e\vec{A}^{(4)}\right)^2 = -m_0^2 c^2}} \quad , \quad \vec{A}^{(4)} = \left(\vec{A}, \phi\right)$$

$$\text{thereby} \quad 0 = \frac{\partial}{\partial q^\nu}\left(p_\mu - eA_\mu\right) \rightarrow \frac{\partial}{\partial q^\nu} p_\mu = e \frac{\partial}{\partial q^\nu} A_\mu$$

$$\text{and} \quad \Box p_\mu = e \Box A_\mu \propto j_\mu$$

(6-5)

The potential equations of electrodynamics developed here from the momentum gradients are equivalent to Maxwell's equations. The electromagnetic fields $\vec{E}$ and $\vec{B}$ must be considered as derivatives of the potentials for this purpose.

$$\text{Potential equations:} \quad \Box A_\mu = \frac{1}{c} j_\mu , \qquad \vec{j}^{(4)} = \rho \cdot \left(\vec{v}, c\right) = \rho_0 \underbrace{\vec{v}^{(4)}}_{v^\mu \vec{g}_\mu^{(4)}}$$

$$\text{Equivalent to Maxwell's equations with:} \quad \begin{cases} \vec{E} \equiv -\vec{\nabla}\phi - \frac{\partial}{\partial ct}\vec{A} \\ \vec{B} \equiv \vec{\nabla} \times \vec{A} \end{cases}$$

(6-6)

Example: Stationary point source at rest

The Coulomb potential or gravitational potential results immediately for a stationary point source.

The source emits the ability to change the momentum. This ability is distributed on the spherical surface $4\pi r^2$ around the source, and is thus inversely proportional to the square of the distance to the source, just as Newton formulated it in his law of gravity.

The difference between the various interactions is the number of charge types. In gravity it is one: the mass. In electromagnetism there are positive and negative charges, and in strong interaction there are three colors of quarks.

$$\text{Potential equation:} \quad \Box A_\mu = \frac{1}{c} j_\mu$$

$$\text{Stationary:} \quad \begin{cases} \Box A_\mu \to \Delta A_\mu \\ j^\mu = \rho \cdot (\vec{v}, c) \to \rho \cdot (\vec{0}, c) \end{cases}$$

$$\text{Point charge at rest:} \quad j^\mu(\vec{r}) \to j^4(\vec{r}) \to ec\delta(\vec{r})$$

$$\text{Hence:} \quad \Delta\phi(\vec{r}) = \underbrace{g_{44}}_{-1} e\delta(\vec{r}), \qquad \underbrace{\int_{\text{Sphere}} \Delta\phi d^3r}_{\int_{\partial\text{Sphere}} \vec{\nabla}\phi \cdot d^2\vec{f} = 4\pi r^2 \hat{e}_r \vec{\nabla}\phi} = -e \underbrace{\int_{\text{Sphere}} \delta(\vec{r}) d^3r}_{1}$$

$$\text{Coulomb potential, gravitational potential:} \quad \vec{\nabla}\phi = -\frac{e}{4\pi r^2}\hat{e}_r, \quad \phi = \frac{e}{4\pi r}$$

(6-7)

General solution for stationary currents

Due to the linearity of the potential equations, the sources can be integrated directly. This is simple for the boundary conditions of free space $\left(\vec{r} \to \infty : \quad A_\mu \to 0\right)$.

$$\text{Stationary:} \quad \Box A_\mu = \frac{1}{c} j_\mu \quad \to \quad \Delta A_\mu = \frac{1}{c} j_\mu$$

$$\text{Vektor potential:} \quad \Delta\vec{A}(\vec{r}) = \frac{1}{c}\vec{j}(\vec{r})$$

$$\text{General solution:} \quad \vec{A}(\vec{r}) = -\frac{1}{4\pi c}\int \vec{j}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}d^3r'$$

$$\text{Prove:} \quad \text{with} \quad \Delta\frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\delta(\vec{r}-\vec{r}')$$

$$\text{is} \quad \Delta\vec{A}(\vec{r}) = -\frac{1}{4\pi c}\int \vec{j}(\vec{r}')\underbrace{\Delta\frac{1}{|\vec{r}-\vec{r}'|}}_{-4\pi\delta(\vec{r}-\vec{r}')}d^3r' = \frac{1}{c}\vec{j}(\vec{r})$$

(6-8)

Equation of motion with field A_μ

The classical equations of motion of a charged particle in the electromagnetic field are given in Hamiltonian mechanics:

$$\text{Hamiltonian:} \quad H = \frac{1}{2m}\left(p^\mu - eA^\mu\right)\left(p_\mu - eA_\mu\right) = \frac{1}{2m}\left(p^\mu p_\mu - 2ep^\mu A_\mu + O\left(e^2\right)\right)$$

$$\text{Equation of motion:} \quad \frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial q^\mu} = \frac{e}{m} \underbrace{\frac{\partial A_\nu}{\partial q^\mu}}_{\substack{F_{\mu\nu} \\ \text{Field tensor}}} p^\nu = eF_{\mu\nu} v^\nu$$

(6-9)

Free motion in Riemannian space

Equations of motion also arise in a space without potential A_μ if this space is curved. The forces then result from the inertia of the body. The space curvature is described by the metric. The metric and their derivatives are summarized in the Christoffel symbol.

$$\text{Acceleration:} \quad 0 = \frac{d^2\vec{r}^{(4)}}{d\tau^2} = \frac{d}{d\tau}\frac{\partial \vec{r}^{(4)}}{\partial q^\mu}\frac{dq^\mu}{d\tau} = \underbrace{\left(\frac{\partial}{\partial q^\nu}\frac{\partial \vec{r}^{(4)}}{\partial q^\mu}\right)}_{\frac{\partial}{\partial q^\nu}\vec{g}_\mu^{(4)}}\frac{dq^\mu}{d\tau}\frac{dq^\nu}{d\tau} + \underbrace{\frac{\partial \vec{r}^{(4)}}{\partial q^\mu}}_{\vec{g}_\mu^{(4)}}\frac{d^2 q^\mu}{d\tau^2}$$

$$\text{Hence:} \quad \vec{g}_\mu^{(4)}\frac{d^2 q^\mu}{d\tau^2} = -\frac{\partial \vec{g}_\mu^{(4)}}{\partial q^\nu}\frac{dq^\mu}{d\tau}\frac{dq^\nu}{d\tau} \quad \text{or} \quad \frac{dv^\lambda}{d\tau} = -\underbrace{\overbrace{\Gamma^\lambda_{\mu\nu}}^{\equiv\, \vec{g}^{(4)\lambda}\cdot\frac{\partial \vec{g}_\mu^{(4)}}{\partial q^\nu}}}_{\substack{\text{Christoffel} \\ \text{symbols}}} v^\mu v^\nu$$

$$\text{Equations of m.:} \quad \frac{d}{d\tau}p_\mu = -m_0 g_{\mu\lambda}\Gamma^\lambda_{\kappa\nu}v^\kappa v^\nu \qquad \underbrace{\text{Comparisons: } F_z = \frac{m}{R}v^2}_{\text{Centrifugal force on circular path}}$$

Generally, the centrifugal forces are proportional to the velocity square, as in the usual centrifugal force of a circular motion.

Equivalence principle

The observed quantity is always the momentum gradient. It is indistinguishable whether it is caused by a field or by inertial forces. This can be used to relate field and metric.

$$\text{Equations of motion:} \quad \frac{d}{d\tau}p_\mu = \underbrace{e\frac{\partial A_\nu}{\partial q^\mu}v^\nu}_{\text{Flat space with gravitational field}} = \underbrace{-m_0 g_{\mu\lambda}\Gamma^\lambda_{\kappa\nu}v^\kappa v^\nu}_{\text{Curved space without field:}}$$

$$\text{Equivalence:} \quad e\frac{\partial A_\nu}{\partial q^\mu}v^\nu = -m_0 g_{\mu\lambda}\Gamma^\lambda_{\kappa\nu}v^\kappa v^\nu$$

$$\text{or:} \quad \underbrace{\underbrace{e\frac{\partial A_\nu}{\partial q^\mu}}_{=\frac{\partial p_\nu}{\partial q^\mu}} = -m_0 g_{\mu\lambda}\Gamma^\lambda_{\kappa\nu}v^\kappa}_{\text{Equation for the metric}}$$

(6-10)

The metric can thereby be determined from the continuity of the momentum gradients. For a resting point charge, the curvature of the space around a black hole in the form of the Schwarzschild metric is provided as an example in Chap06_5.

Christoffel symbols and metric

The Christoffel symbols can be determined solely from the metric $g_{\mu\nu}$ without explicit knowledge of the ground vectors $\vec{g}_{\mu}^{(4)}$:

$$\text{Christoffel symbols:} \quad \Gamma_{\mu\nu}^{\lambda} \equiv \vec{g}^{(4)\lambda} \cdot \frac{\partial \vec{g}_{\mu}^{(4)}}{\partial q^{\nu}}$$

$$\text{with:} \quad \frac{\partial g_{\kappa\mu}}{\partial q^{\nu}} = \frac{\partial \vec{g}_{\mu}^{(4)}}{\partial q^{\nu}} \cdot \vec{g}_{\kappa}^{(4)} + \frac{\partial \vec{g}_{\kappa}^{(4)}}{\partial q^{\nu}} \cdot \vec{g}_{\mu}^{(4)} = \frac{\partial^2 \vec{r}^{(4)}}{\partial q^{\nu} \partial q^{\mu}} \cdot \frac{\partial \vec{r}^{(4)}}{\partial q^{\kappa}} + \frac{\partial^2 \vec{r}^{(4)}}{\partial q^{\kappa} \partial q^{\nu}} \cdot \frac{\partial \vec{r}^{(4)}}{\partial q^{\mu}}$$

$$\text{follows:} \quad \underbrace{\frac{\partial g_{\mu\kappa}}{\partial q^{\nu}}}_{\cancel{\frac{\partial^2 \vec{r}^{(4)}}{\partial q^{\kappa} \partial q^{\nu}} \cdot \frac{\partial \vec{r}^{(4)}}{\partial q^{\mu}}} + \frac{\partial^2 \vec{r}^{(4)}}{\partial q^{\mu} \partial q^{\nu}} \cdot \frac{\partial \vec{r}^{(4)}}{\partial q^{\kappa}}} + \underbrace{\frac{\partial g_{\kappa\nu}}{\partial q^{\mu}}}_{\cancel{\frac{\partial^2 \vec{r}^{(4)}}{\partial q^{\kappa} \partial q^{\mu}} \cdot \frac{\partial \vec{r}^{(4)}}{\partial q^{\nu}}} + \frac{\partial^2 \vec{r}^{(4)}}{\partial q^{\mu} \partial q^{\nu}} \cdot \frac{\partial \vec{r}^{(4)}}{\partial q^{\kappa}}} - \underbrace{\frac{\partial g_{\mu\nu}}{\partial q^{\kappa}}}_{\cancel{\frac{\partial^2 \vec{r}^{(4)}}{\partial q^{\kappa} \partial q^{\nu}} \cdot \frac{\partial \vec{r}^{(4)}}{\partial q^{\mu}}} + \cancel{\frac{\partial^2 \vec{r}^{(4)}}{\partial q^{\kappa} \partial q^{\mu}} \cdot \frac{\partial \vec{r}^{(4)}}{\partial q^{\nu}}}} = 2 \frac{\partial^2 \vec{r}^{(4)}}{\partial q^{\mu} \partial q^{\mu}} \cdot \frac{\partial \vec{r}^{(4)}}{\partial q^{\kappa}} = 2 \frac{\partial \vec{g}_{\mu}^{(4)}}{\partial q^{\nu}} \cdot \vec{g}_{\kappa}^{(4)}$$

$$= 2 \frac{\partial \vec{g}_{\mu}^{(4)}}{\partial q^{\nu}} \cdot \vec{g}^{(4)\lambda} g_{\lambda\kappa} = 2 g_{\kappa\lambda} \Gamma_{\mu\nu}^{\lambda}$$

$$\text{so:} \quad \Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\kappa} \left(\frac{\partial g_{\kappa\mu}}{\partial q^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial q^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial q^{\kappa}} \right)$$

(6-11)