EBERHARD KARLS UNIVERSITÄT TÜBINGEN

MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT
Fachbereich Physik

Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

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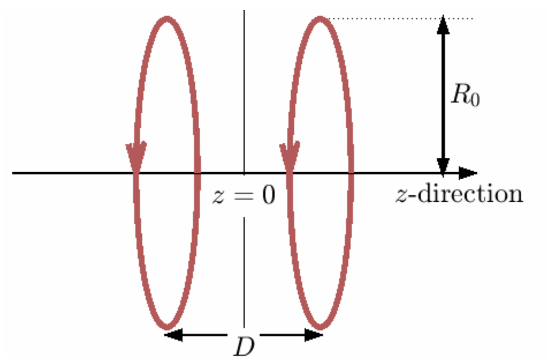
6.2 Helmholtz coil (Computational example)

A Helmholtz coil arrangement consists of two current-carrying conductor rings in the same direction on a common axis. For an optimal homogeneous magnetic field between the coils, we must choose the coil radius and distance equal $(R_0 = D)$.



We calculate the magnetic field lines in the *x*-*z* plane, plot them, and compare them to the experiment.

Vector potential: $\vec{A}\left(\vec{R}\right) = \frac{1}{c}\int \vec{j}\left(\vec{r}\right)\frac{1}{\left|\vec{R} - \vec{r}\right|}d^3r$

Current: $\vec{j}\left(\vec{r}\right) = J\hat{e}_\varphi\delta(r - R_0)\left(\delta\left(z - \frac{R_0}{2}\right) + \delta\left(z + \frac{R_0}{2}\right)\right)$

Parameter: $R_0 = J = c = 1$

```
clear all
syms z  J G jv
syms r R R0 Z phi Phi  positive
Par=[R0==1 J==1]
```

Par = $\left(R_0 = 1 \quad J = 1\right)$

1 Basic vectors in cylinder coordinates

$$\vec{r} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}, \ \vec{R} = \begin{pmatrix} R\cos\Phi \\ R\sin\Phi \\ Z \end{pmatrix}, \ \hat{e}_\varphi = \frac{1}{r}\frac{\partial}{\partial\varphi}\vec{r}$$

```
rv=[r*cos(phi);r*sin(phi);z]
```

rv =

$$\begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{pmatrix}$$

```
Rv=[R*cos(Phi);R*sin(Phi);Z]
```

Rv =

$$\begin{pmatrix} R\cos(\Phi) \\ R\sin(\Phi) \\ Z \end{pmatrix}$$

```
e_phi=diff(rv,phi)/r
```

e_phi =

$$\begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}$$

2 Vector potential

Vector potential: $\vec{A}\left(\vec{R}\right) = \frac{1}{c}\int \vec{j}\,(\vec{r})\,\frac{1}{\left|\vec{R} - \vec{r}\right|}\,rdrd\varphi dz$

Current $\vec{j}\left(\vec{r}\right) = J\hat{e}_\varphi\delta(r - R_0)\left(\delta\left(z - \frac{R_0}{2}\right) + \delta\left(z + \frac{R_0}{2}\right)\right)$

```
jv=J*e_phi*dirac(r-R0)*(dirac(z-R0/2)+dirac(z+R0/2))
```

jv =

$$\begin{pmatrix} -J\,\sin(\varphi)\,\delta(R_0 - r)\,\left(\delta\left(\frac{R_0}{2} - z\right) + \delta\left(\frac{R_0}{2} + z\right)\right) \\ J\,\cos(\varphi)\,\delta(R_0 - r)\,\left(\delta\left(\frac{R_0}{2} - z\right) + \delta\left(\frac{R_0}{2} + z\right)\right) \\ 0 \end{pmatrix}$$

Propagator $G = \frac{1}{|\vec{r} - \vec{R}|}$

```
G=1/simplify(norm(rv-Rv))
```

G =

$$\frac{1}{\sqrt{|Z-z|^2 + (R\cos(\Phi) - r\cos(\varphi))^2 + (R\sin(\Phi) - r\sin(\varphi))^2}}$$

Potential without φ integration $\vec{A}^{(\varphi)} = \int \vec{j}\, G r dr dz,\ \vec{A} = \int \vec{A}^{(\varphi)} d\varphi$

```
Av_phi=int(int(r*jv*G,r,0,inf),z,-inf,inf)
```

Av_phi =

$$\begin{pmatrix} -\frac{J R_0 \sin(\varphi)}{\sigma_1} - \frac{J R_0 \sin(\varphi)}{\sigma_2} \\ \frac{J R_0 \cos(\varphi)}{\sigma_1} + \frac{J R_0 \cos(\varphi)}{\sigma_2} \\ 0 \end{pmatrix}$$

where

$$\sigma_1 = \sqrt{\left|\frac{R_0}{2} - Z\right|^2 + (R\cos(\Phi) - R_0\cos(\varphi))^2 + (R\sin(\Phi) - R_0\sin(\varphi))^2}$$

$$\sigma_2 = \sqrt{(R\cos(\Phi) - R_0\cos(\varphi))^2 + (R\sin(\Phi) - R_0\sin(\varphi))^2 + \left(\frac{R_0}{2} + Z\right)^2}$$

in the *x*-*z* plane $(\Phi = 0)$ and with parameters:

```
Av_phi=sube(Av_phi,[Phi==0 Par])
```

Av_phi =

$$\begin{pmatrix} -\frac{\sin(\varphi)}{\sigma_1} - \frac{\sin(\varphi)}{\sigma_2} \\ \frac{\cos(\varphi)}{\sigma_2} + \frac{\cos(\varphi)}{\sigma_1} \\ 0 \end{pmatrix}$$

where

$$\sigma_1 = \sqrt{\sin(\varphi)^2 + (R - \cos(\varphi))^2 + \left|Z - \frac{1}{2}\right|^2}$$

$$\sigma_2 = \sqrt{\left(Z + \frac{1}{2}\right)^2 + \sin(\varphi)^2 + (R - \cos(\varphi))^2}$$

3 x and z component of the vector potential

$$A_x = \int_0^{2\pi} A_1{}^{(\varphi)} d\varphi, \quad A_z = \int_0^{2\pi} A_3{}^{(\varphi)} d\varphi$$

```
A_x=int(Av_phi(1),phi,0,2*pi)
```

A_x = 0

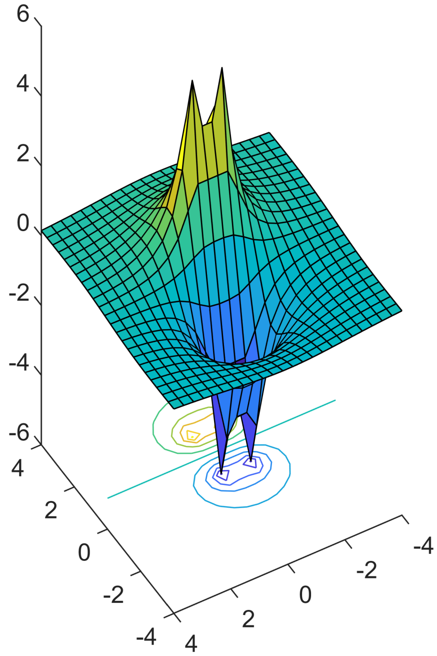
```
A_z=int(Av_phi(3),phi,0,2*pi)
```

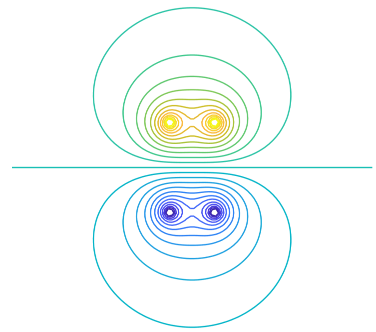
A_z = 0

4 y component A_y with field lines (see also appendix below)

$$A_y = \int_0^{2\pi} A_2{}^{(\varphi)} d\varphi$$

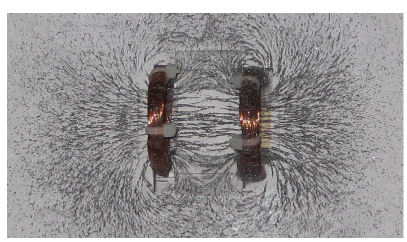
```
[Rn,Zn,Ayn]=GetAy(Av_phi(2),24);
subplot(1,2,1)
hold on
surfc(Rn,Zn,Ayn);
axis([-4 4 -4 4 -6 6])
view([-120.2 25.0])

[Rn,Zn,Ayn]=GetAy(Av_phi(2),100);
subplot(1,2,2)
contour(Zn,Rn,Ayn,6*linspace(-1,1,25))
axis equal
axis off
```





Comparison with the experiment:

Helmholtz coil and iron filings on a glass plate

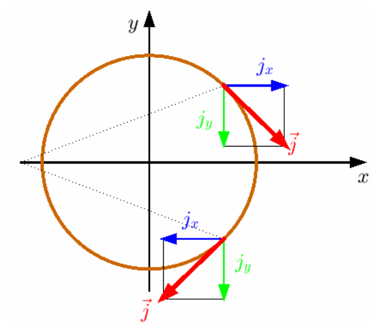


Appendix: Field lines for Helmholtz coil in the z-x plane

Vector potential: $\vec{A}\left(\vec{R}\right) = \frac{1}{c}\int \vec{j}\left(\vec{r}\right) \frac{1}{\left|\vec{R} - \vec{r}\right|} d^3r$

Current: $\vec{j}\left(\vec{r}\right) = J\hat{e}_\varphi \delta(r - R_0)\left(\delta\left(z - \frac{R_0}{2}\right) + \delta\left(z + \frac{R_0}{2}\right)\right)$

Helmholtz coil and current components j:

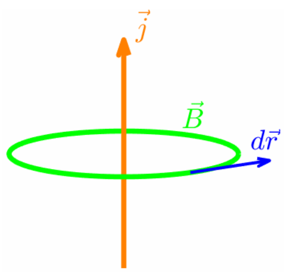


From symmetry considerations follows $A_x = A_z = 0$:

$$\text{Vector potential } z\text{-}x\text{-plane:}\quad \underbrace{\vec{A}(x,0,z)=A_y(x,z)\hat{e}_y}_{\substack{A_z(x,z)=0:\quad \vec{j}\text{ has no z-component!}\\ A_x(x,z)=0:\quad j_x(x,y,z)=-j_x(x,-y,z)}}$$

The magnetic field lines in the *z*-*x* plane can be represented as contour lines of the vector potential A_y:

Field lines:



$$\overbrace{0=\vec{B}\times d\vec{r}}^{\text{Conditional equations}}=\begin{pmatrix}B_y dz-B_z dy\\ B_z dx-B_x dz\\ B_x dy-B_y dx\end{pmatrix}=\begin{pmatrix}\left(\cancel{\frac{\partial A_x}{\partial z}}-\cancel{\frac{\partial A_z}{\partial x}}\right)dz-\left(\frac{\partial A_y}{\partial x}-\cancel{\frac{\partial A_x}{\partial y}}\right)\cancel{dy}\\ \left(\frac{\partial A_y}{\partial x}-\cancel{\frac{\partial A_x}{\partial y}}\right)dx-\left(\cancel{\frac{\partial A_z}{\partial y}}-\frac{\partial A_y}{\partial z}\right)dz\\ \left(\cancel{\frac{\partial A_z}{\partial y}}-\frac{\partial A_y}{\partial z}\right)\cancel{dy}-\left(\cancel{\frac{\partial A_x}{\partial z}}-\cancel{\frac{\partial A_z}{\partial x}}\right)dx\end{pmatrix}$$

$$=\left(\frac{\partial A_y(x,z)}{\partial x}dx+\frac{\partial A_y(x,z)}{\partial z}dz\right)\hat{e}_y=dA_y(x,z)\hat{e}_y=0$$

Function for φ-integration (numerical)

```
function [Rn,Zn,Ayn]=GetAy(Ay_phi,N)
[Rn,Zn]=meshgrid(linspace(-4,4,N));
Ay_phim=matlabFunction(Ay_phi);
Ayn=integral(@(phi)Ay_phim(Rn,Zn,phi),0,2*pi,'ArrayValued',true);
end
```