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Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

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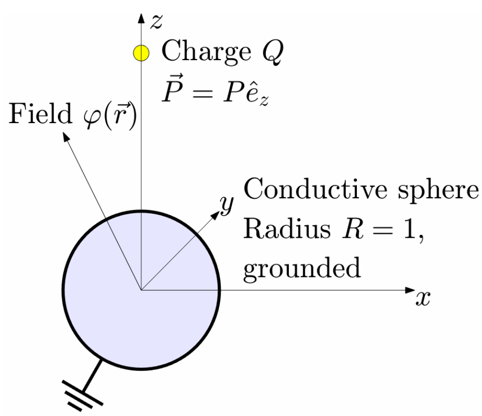
6.3 Boundary value problem: charge over grounded sphere (Computational example)

A charge Q is located above a grounded metal sphere. The electrical potential can be calculated with the help of a mirror charge q (Method of Images).



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Parameter: $Q = 1,\ P = 3.5$

```
syms r theta P p Q q x z real
Par=[Q==1,P==3.5]
```

Par =

$$\left(Q = 1 \quad P = \frac{7}{2}\right)$$

1 Coordinates

Position vector in the x-z plane $(\varphi = 0)$: $\vec{r} = \begin{pmatrix} r\sin\theta \\ 0 \\ r\cos\theta \end{pmatrix}$

```
r_v=[r*sin(theta);0;r*cos(theta)]
```

r_v =

$$\begin{pmatrix} r\sin(\theta) \\ 0 \\ r\cos(\theta) \end{pmatrix}$$

Positions of charge Q and mirror charge q

with $\hat{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$:

Position of charge Q: $\vec{P} = P\hat{e}_z$

Position of mirror charge q: $\vec{p} = p\hat{e}_z$

```
e_z=[0;0;1];
P_v=P*e_z
```

P_v =

$$\begin{pmatrix} 0 \\ 0 \\ P \end{pmatrix}$$

```
p_v=p*e_z
```

p_v =

$$\begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}$$

2 Contributions to the potential

Potential of the charge: $\varphi_Q(\vec{r}) = \dfrac{Q}{\left|\vec{r} - P\hat{e}_z\right|}$

```
phi_Q=simplify(Q/norm(P_v-r_v))
```

phi_Q =

$$\frac{Q}{\sqrt{P^2 - 2\cos(\theta)\, P\, r + r^2}}$$

Potential of the mirror charge $\varphi_q(\vec{r}\,) = \frac{q}{|\vec{r} - p\hat{e}_z|}$

```
phi_q=simplify(q/norm(p_v-r_v))
```

phi_q =

$$\frac{q}{\sqrt{p^2 - 2\cos(\theta)\, p\, r + r^2}}$$

Total potential: $\phi = \phi_Q + \phi_q \;\; (\forall r > 1)$

```
phi=phi_Q+phi_q
```

phi =

$$\frac{q}{\sqrt{p^2 - 2\cos(\theta)\, p\, r + r^2}} + \frac{Q}{\sqrt{P^2 - 2\cos(\theta)\, P\, r + r^2}}$$

3 Boundary conditions

Boundary conditions: $\varphi|_{r=1} = 0 \;\; \rightarrow \;\; p(Q,P),\; q(Q,P)$

```
sube(phi,r==1)==0
```

ans =

$$\frac{q}{\sqrt{p^2 - 2\cos(\theta)\, p + 1}} + \frac{Q}{\sqrt{P^2 - 2\cos(\theta)\, P + 1}} = 0$$

```
isolate(ans,cos(theta))
```

ans =

$$\cos(\theta) = -\frac{-P^2\, q^2 + Q^2\, p^2 + Q^2 - q^2}{2\, P\, q^2 - 2\, Q^2\, p}$$

For this equation to hold for all $\theta \in [0, \pi]$, the numerator and denominator of the right side must be zero!

```
[A,B]=numden(rhs(ans));
assume(Q>0 & q<0 & P>1 & p<1)
solve([A,B],[q,p]);
[p==ans.p,q==ans.q]
```

ans =

$$\left(p = \frac{1}{P} \quad q = -\frac{Q}{P}\right)$$

```
Phi=simplify(sube(phi,ans))
```

```
Phi =
```

$$\frac{Q}{\sqrt{P^2 - 2\cos(\theta)\,P\,r + r^2}} - \frac{Q}{\sqrt{P^2\,r^2 - 2\cos(\theta)\,P\,r + 1}}$$

4 Potential in Cartesian coordinates

Transformation: $\cos\theta = \frac{z}{r},\ r = \sqrt{x^2 + z^2}$

```
Phi_xy=sube(Phi,[Par,cos(theta)==z/r r==sqrt(x^2+z^2)])
```

```
Phi_xy =
```

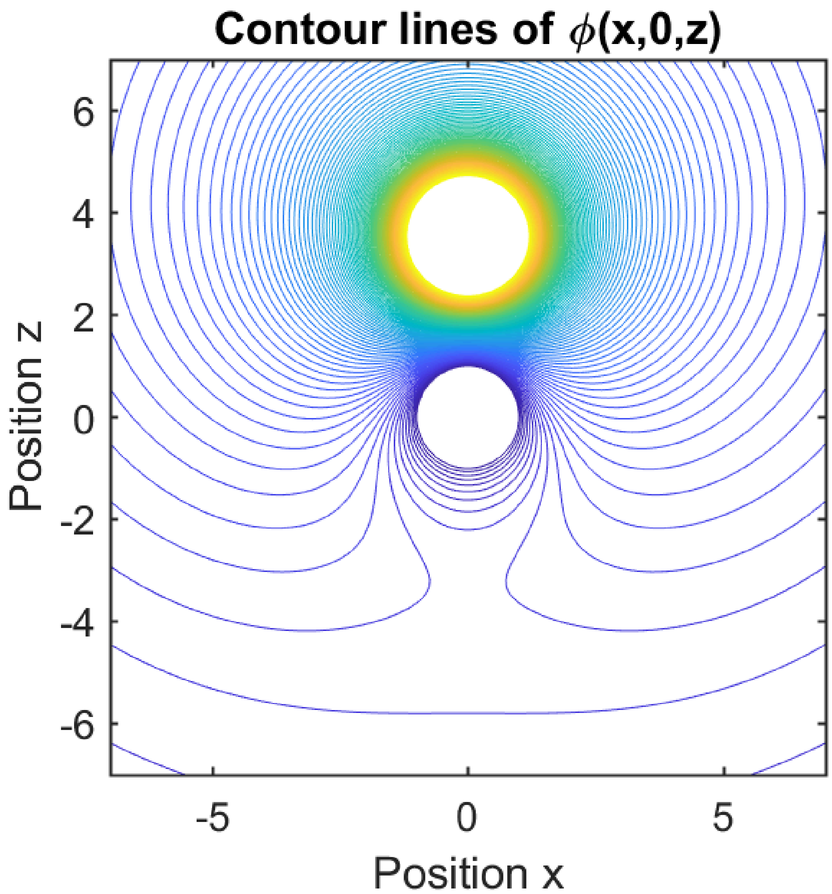
$$\frac{1}{\sqrt{x^2 + z^2 - 7\,z + \frac{49}{4}}} - \frac{1}{\sqrt{\frac{49\,x^2}{4} + \frac{49\,z^2}{4} - 7\,z + 1}}$$

5 Contour lines of the potential

```
N=100;
[XX,ZZ]=meshgrid(linspace(-7,7,N),linspace(-7,7,N));
F=matlabFunction(Phi_xy)
```

```
F = function_handle with value:
    @(x,z)-1.0./sqrt(z.*-7.0+x.^2.*(4.9e1./4.0)+z.^2.*(4.9e1./4.0)+1.0)+1.0./sqrt(z.*-7.0+x.^2+z.^2+4.9e1./4.0)
```

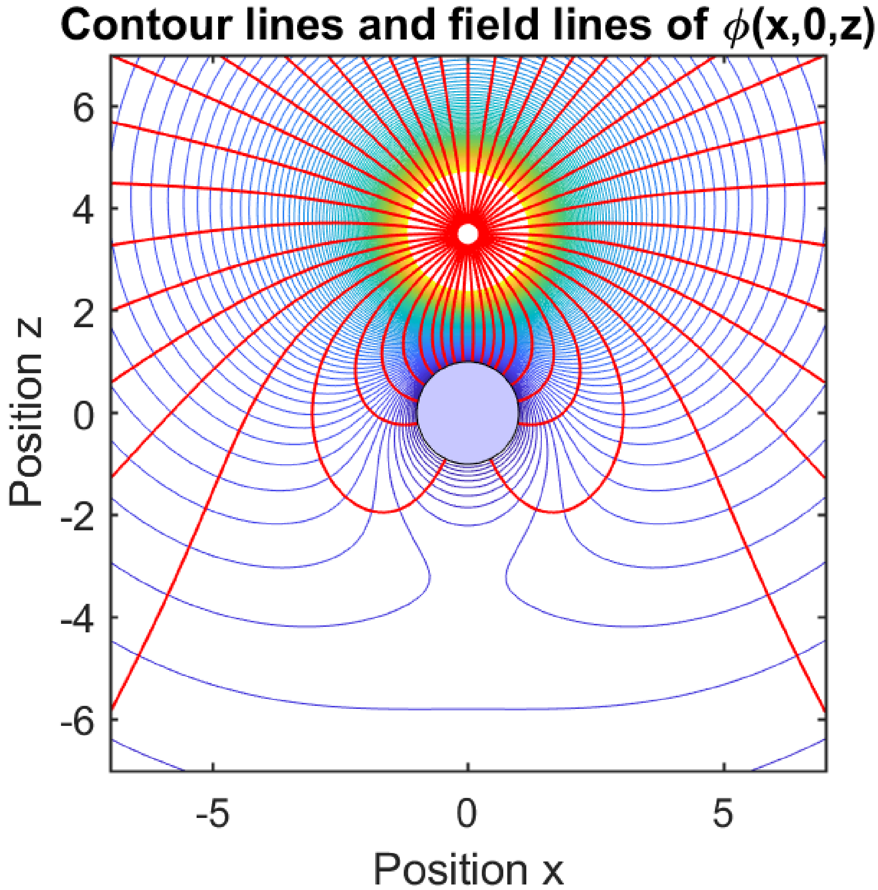
```
contour(XX,ZZ,F(XX,ZZ),linspace(0,0.75,100))
axis equal
set(0,'defaultaxesfontsize',14,'defaultaxeslinewidth',1)
xlabel('Position x')
ylabel('Position z')
title('Contour lines of \phi(x,0,z)')
```



6 Electric field lines

Electric field in the presentation plane $\begin{pmatrix} E_x \\ E_y \end{pmatrix} = -\begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{pmatrix}$

```
Ex=matlabFunction(-diff(Phi_xy,x));
Ez=matlabFunction(-diff(Phi_xy,z));
r0=0.2;
phi0=linspace(0,2*pi,50);
h=streamline(XX,ZZ,Ex(XX,ZZ),Ez(XX,ZZ),...
    r0*sin(phi0),r0*cos(phi0)+double(sube(P,Par)));
set(h,'Color','r','LineWidth',1)
rectangle('Position',[-1 -1 2 2],'FaceColor',[.8 .8 1],'Curvature',[1 1])
title('Contour lines and field lines of \phi(x,0,z)')
```



Contour lines and field lines of ϕ(x,0,z)
6
4
2
0
-2
-4
-6
Position z
-5
0
5
Position x