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MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT
Fachbereich Physik

Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

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6.4 Radiation of the Hertzian dipole (Computational example)

A Hertzian dipole consists of a negative point charge at the coordinate origin and a harmonic-moving positive point charge at $\vec{a}(t)$.

Trajectory of the positive charge: $\vec{a}(t) = a_0 \cos(\omega t)\hat{e}_z$

Charge density: $\rho\left(\vec{r}, t\right) = q\left(\delta\left(\vec{r} - \vec{a}(t)\right) - \delta\left(\vec{r}\right)\right)$

Current: $\vec{j}\left(\vec{r}, t\right) = \vec{v}\left(\vec{r}, t\right)\rho\left(\vec{r}, t\right) = q\dot{\vec{a}}(t)\delta\left(\vec{r} - \vec{a}(t)\right)$

Vector potential: $\vec{A}\left(\vec{r}, t\right) = \frac{\mu_0}{4\pi}\int dt' d^3\vec{r}\,' \frac{\vec{j}\left(\vec{r}\,', t'\right)\delta\left(t - c^{-1}\left|\vec{r} - \vec{r}\,'\right| - t'\right)}{\left|\vec{r} - \vec{r}\,'\right|}$

Parameter: $a_0 = q = \mu = \omega = c = 1$

```
clear all
syms r t tb a0 q av positive
syms theta phi omega mu c x z real

Par=[a0==1 q==1 mu==1 omega==1 c==1]
```

Par = $\left(a_0 = 1 \quad q = 1 \quad \mu = 1 \quad \omega = 1 \quad c = 1\right)$

1 Spherical coordinates

Spherical coordinates: $\vec{r} = \begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix}$

Jacobian: $J = \left(\frac{\partial x_i}{\partial q_j}\right) = \left(\vec{g}_r, \vec{g}_\theta, \vec{g}_\varphi\right), \quad \rightarrow \quad \hat{e}_r = \frac{\vec{g}_r}{\left|\vec{g}_r\right|}, \hat{e}_\theta = \frac{\vec{g}_\theta}{\left|\vec{g}_\theta\right|}, \hat{e}_\varphi = \frac{\vec{g}_\varphi}{\left|\vec{g}_\varphi\right|}$

z-direction: $\hat{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

```
r_v=[r*sin(theta)*cos(phi);r*sin(theta)*sin(phi);r*cos(theta)]
```

r_v =

$$\begin{pmatrix} r\cos(\varphi)\sin(\theta) \\ r\sin(\varphi)\sin(\theta) \\ r\cos(\theta) \end{pmatrix}$$

```
J=jacobian(r_v,[r,theta,phi]);
e_r=J(:,1)
```

e_r =

$$\begin{pmatrix} \cos(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\theta) \end{pmatrix}$$

```
e_theta=J(:,2)/r
```

e_theta =

$$\begin{pmatrix} \cos(\varphi)\cos(\theta) \\ \cos(\theta)\sin(\varphi) \\ -\sin(\theta) \end{pmatrix}$$

```
e_phi=J(:,3)/(r*sin(theta))
```

e_phi =

$$\begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}$$

```
E_z_xA=[0;0;1]
```

```
E_z_xA = 3×1
     0
     0
     1
```

```
XPlane=[phi==0 theta==pi/2 r==x]
```

XPlane =

$$\left(\varphi = 0 \quad \theta = \frac{\pi}{2} \quad r = x\right)$$

```
XZPlane=[phi==0 sin(theta)==x/r cos(theta)==z/r r==sqrt(x^2+z^2)]
```

XZPlane =

$$\left(\varphi = 0 \quad \sin(\theta) = \frac{x}{r} \quad \cos(\theta) = \frac{z}{r} \quad r = \sqrt{x^2 + z^2}\right)$$

2 Vector potential in far field approximation

Current of the Hertzian dipole: $\vec{j}\left(\vec{r}, t\right) = q\dot{\vec{a}}(t)\delta\left(\vec{r} - \vec{a}(t)\right) \underset{\text{far field approximation}}{\rightarrow} q\dot{\vec{a}}(t)\delta\left(\vec{r}\right)$

Vector potential: $\vec{A}\left(\vec{r}, t\right) = \frac{\mu_0}{4\pi}\int dt' d^3\vec{r}\,' \frac{\vec{j}\left(\vec{r}\,', t'\right)\delta\left(t - c^{-1}\left|\vec{r} - \vec{r}\,'\right| - t'\right)}{\left|\vec{r} - \vec{r}\,'\right|}$

$$\rightarrow \quad \frac{\mu_0}{4\pi}\int dt' d^3\vec{r}\,' \frac{q\dot{\vec{a}}(t')\delta\left(\vec{r}\,'\right)\delta\left(t - c^{-1}\left|\vec{r} - \vec{r}\,'\right| - t'\right)}{\left|\vec{r} - \vec{r}\,'\right|}$$

$$= \frac{\mu_0}{4\pi r} q\dot{\vec{a}}(t - r/c) = -\frac{\mu_0}{4\pi r} q\omega \sin(t - r/c)\hat{e}_z$$

```
A_v=(-mu*omega*a0*q*sin(omega*(t-r/c))/(4*pi*r))*E_z_xA
```

A_v =

$$\begin{pmatrix} 0 \\ 0 \\ -\frac{a_0\,\mu\,\omega\,q\sin\left(\omega\,\left(t - \frac{r}{c}\right)\right)}{4\,r\,\pi} \end{pmatrix}$$

in spherical coordinates basis: $\vec{A}^{(Ku)} = \begin{pmatrix} \vec{A}\cdot\hat{e}_r \\ \vec{A}\cdot\hat{e}_\vartheta \\ \vec{A}\cdot\hat{e}_\varphi \end{pmatrix}$

```
A_Ku=[dot(A_v,e_r);dot(A_v,e_theta);dot(A_v,e_phi)]
```

A_Ku =

$$\begin{pmatrix} -\frac{a_0\,\mu\,\omega\,q\,\sin\left(\omega\,\left(t-\frac{r}{c}\right)\right)\cos(\theta)}{4\,r\,\pi} \\ \frac{a_0\,\mu\,\omega\,q\,\sin\left(\omega\,\left(t-\frac{r}{c}\right)\right)\sin(\theta)}{4\,r\,\pi} \\ 0 \end{pmatrix}$$

3 *B* and *E* field in spherical coordinates basis

Magnetic field: $\vec{B}^{(Ku)} = rot\ \vec{A}^{(Ku)}$

Electric field: $\vec{E}^{(Ku)} = \int rot\ \vec{B}^{(Ku)}\ dt$

The curl in spherical coordinates is calculated with the function Curl3d at the end of this script.

```
B_Ku=Curl3d(A_Ku,[r,theta,phi],[1,r,r*sin(theta)])
```

B_Ku =

$$\begin{pmatrix} 0 \\ 0 \\ \frac{a_0\,\mu\,\omega\,q\,\sin(\theta)\,\left(c\,\sin\left(\frac{\omega\,(r-c\,t)}{c}\right)-\omega\,r\,\cos\left(\frac{\omega\,(r-c\,t)}{c}\right)\right)}{4\,c\,r^2\,\pi} \end{pmatrix}$$

```
E_Ku=int(Curl3d(B_Ku,[r,theta,phi],[1,r,r*sin(theta)]),t)
```

E_Ku =

$$\begin{pmatrix} \frac{a_0\,c\,\mu\,q\,\cos(\theta)\,\sigma_2+a_0\,\mu\,\omega\,q\,r\,\sigma_1\,\cos(\theta)}{2\,c\,r^3\,\pi} \\ \frac{a_0\,\mu\,q\,\sin(\theta)\,\sigma_2\,(c^2-\omega^2\,r^2)+a_0\,c\,\mu\,\omega\,q\,r\,\sigma_1\,\sin(\theta)}{4\,c^2\,r^3\,\pi} \\ 0 \end{pmatrix}$$

where

$$\sigma_1 = \sin\left(\frac{\omega\,(r-c\,t)}{c}\right)$$

$$\sigma_2 = \cos\left(\frac{\omega\,(r-c\,t)}{c}\right)$$

4 *B* and *E* on the *x*-axis

x-axis: $r = x,\ \theta = \frac{\pi}{2},\ \varphi = 0$

E field: $\overrightarrow{E} = E_r^{(Ku)}\widehat{e}_r + E_\theta^{(Ku)}$

B field: $\overrightarrow{B} = B_\varphi^{(Ku)}\widehat{e}_\varphi$

```
E_v=E_Ku(1)*e_r+E_Ku(2)*e_theta
```

E_v =

$$\begin{pmatrix} \frac{\cos(\varphi)\cos(\theta)\,\sigma_1}{\sigma_3} + \frac{\cos(\varphi)\sin(\theta)\,\sigma_2}{\sigma_4} \\ \frac{\sin(\varphi)\sin(\theta)\,\sigma_2}{\sigma_4} + \frac{\cos(\theta)\sin(\varphi)\,\sigma_1}{\sigma_3} \\ \frac{\cos(\theta)\,\sigma_2}{\sigma_4} - \frac{\sin(\theta)\,\sigma_1}{\sigma_3} \end{pmatrix}$$

where

$$\sigma_1 = a_0\,\mu\,q\sin(\theta)\cos\left(\frac{\omega\,(r - c\,t)}{c}\right)\,(c^2 - \omega^2\,r^2) + a_0\,c\,\mu\,\omega\,q\,r\sin\left(\frac{\omega\,(r - c\,t)}{c}\right)\sin(\theta)$$

$$\sigma_2 = a_0\,c\,\mu\,q\cos(\theta)\cos\left(\frac{\omega\,(r - c\,t)}{c}\right) + a_0\,\mu\,\omega\,q\,r\sin\left(\frac{\omega\,(r - c\,t)}{c}\right)\cos(\theta)$$

$$\sigma_3 = 4\,c^2\,r^3\,\pi$$

$$\sigma_4 = 2\,c\,r^3\,\pi$$

on the *x* axis

```
sube(E_v,XPlane)
```

ans =

$$\begin{pmatrix} 0 \\ 0 \\ \frac{a_0\,\mu\,q\cos\left(\frac{\omega\,(x - c\,t)}{c}\right)\,(\omega^2\,x^2 - c^2) - a_0\,c\,\mu\,\omega\,q\,x\sin\left(\frac{\omega\,(x - c\,t)}{c}\right)}{4\,c^2\,x^3\,\pi} \end{pmatrix}$$

```
E_z_xA=ans(3)
```

E_z_xA =

$$\frac{a_0\,\mu\,q\cos\left(\frac{\omega\,(x-c\,t)}{c}\right)\,(\omega^2\,x^2-c^2)-a_0\,c\,\mu\,\omega\,q\,x\sin\left(\frac{\omega\,(x-c\,t)}{c}\right)}{4\,c^2\,x^3\,\pi}$$

y component of the *B field on the x axis*

```
B_v=B_Ku(3)*e_phi
```

B_v =

$$\begin{pmatrix} -\frac{a_0\,\mu\,\omega\,q\sin(\varphi)\sin(\theta)\,\sigma_1}{4\,c\,r^2\,\pi} \\ \frac{a_0\,\mu\,\omega\,q\cos(\varphi)\sin(\theta)\,\sigma_1}{4\,c\,r^2\,\pi} \\ 0 \end{pmatrix}$$

where

$$\sigma_1 = c\sin\left(\frac{\omega\,(r-c\,t)}{c}\right)-\omega\,r\cos\left(\frac{\omega\,(r-c\,t)}{c}\right)$$

```
sube(B_v,XPlane)
```

ans =

$$\begin{pmatrix} 0 \\ \frac{a_0\,\mu\,\omega\,q\,\left(c\sin\left(\frac{\omega\,(x-c\,t)}{c}\right)-\omega\,x\cos\left(\frac{\omega\,(x-c\,t)}{c}\right)\right)}{4\,c\,x^2\,\pi} \\ 0 \end{pmatrix}$$

```
B_y_xA=ans(2)
```

B_y_xA =

$$\frac{a_0\,\mu\,\omega\,q\,\left(c\sin\left(\frac{\omega\,(x-c\,t)}{c}\right)-\omega\,x\cos\left(\frac{\omega\,(x-c\,t)}{c}\right)\right)}{4\,c\,x^2\,\pi}$$

5 Graphics

```
R=linspace(0,20);
subplot(2,2,1)
hold on
plot3([0 20],[0 0],[0 0],':k')
B_y_xAM=matlabFunction(sube(B_y_xA,Par))
```

B_y_xAM = function_handle with value:

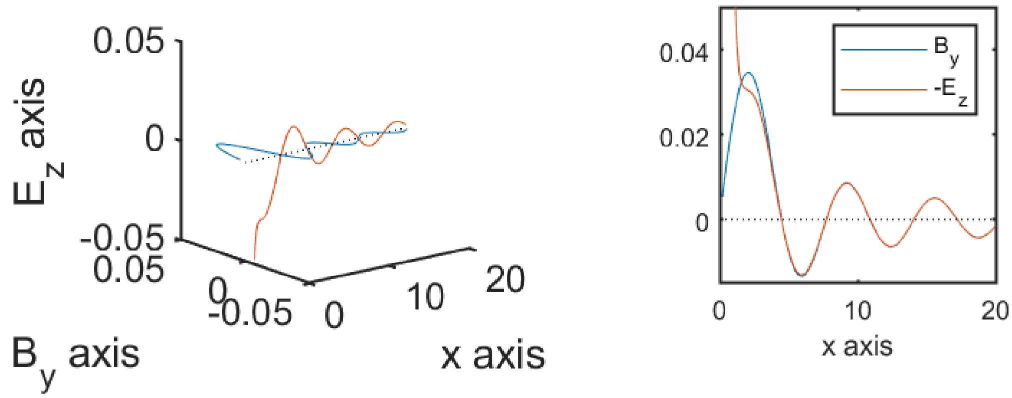
```
    @(t,x)(1.0./x.^2.*(sin(t-x)+x.*cos(t-x)).*(-1.0./4.0))./pi
```

```
plot3(R,B_y_xAM(0,R),0*R)
E_z_xAM=matlabFunction(sube(E_z_xA,Par))
```

```
E_z_xAM = function_handle with value:
    @(t,x)(1.0./x.^3.*(cos(t-x).*(x.^2-1.0)+x.*sin(t-x)).*(1.0./4.0))./pi
```

```
plot3(R,0*R,E_z_xAM(0,R))
axis([0 20 -.05 .05 -.05 .05])
view([-74 23])
axis square
xlabel('x axis')
ylabel('B_y axis')
zlabel('E_z axis')
hold off
subplot(2,2,2)
plot(R,B_y_xAM(0,R),R,-E_z_xAM(0,R),[0 20],[0 0],':k')
axis([0 20 -0.015 0.05])
xlabel('x axis')
axis square
legend('B_y','-E_z')

subplot(2,2,1)
view([-38.673 15.726])
subplot(2,2,2)
view([0.00 90.00])
```



6 Field lines of the *E* field in the *x-z* plane

The electric field lines can be plotted in the *x-z* plane as contour lines of the magnetic field integrated over time:

in the $x-z$ plane: $B_x = 0, B_z = 0,$

Field lines:

$$0 = \underbrace{\vec{E} \times d\vec{r}}_{\left(\int rot\vec{B}\,dt\right)\times d\vec{r}} = \int \begin{pmatrix} \cancel{\frac{\partial B_z}{\partial y}} - \frac{\partial B_y}{\partial z} \\ \cancel{\frac{\partial B_x}{\partial z}} - \cancel{\frac{\partial B_z}{\partial x}} \\ \frac{\partial B_y}{\partial x} - \cancel{\frac{\partial B_x}{\partial y}} \end{pmatrix} dt \times \begin{pmatrix} dx \\ 0 \\ dz \end{pmatrix} = \int \begin{pmatrix} 0 \\ \left(\frac{\partial B_y}{\partial x} + \frac{\partial B_y}{\partial z}\right) dz \\ 0 \end{pmatrix} dt = d \int \vec{B} dt \Big|_{x-z \text{ plane}}$$

```
B_v
```

```
B_v =
```

$$\begin{pmatrix} -\frac{a_0\,\mu\,\omega\,q\sin(\varphi)\sin(\theta)\,\sigma_1}{4\,c\,r^2\,\pi} \\ \frac{a_0\,\mu\,\omega\,q\cos(\varphi)\sin(\theta)\,\sigma_1}{4\,c\,r^2\,\pi} \\ 0 \end{pmatrix}$$

where

$$\sigma_1 = c\sin\left(\frac{\omega\,(r-c\,t)}{c}\right) - \omega\,r\cos\left(\frac{\omega\,(r-c\,t)}{c}\right)$$

```
sube(B_v,XZPlane)
```

```
ans =
```

$$\begin{pmatrix} 0 \\ -\frac{a_0\,\mu\,\omega\,q\,x\,\left(c\sin(\sigma_1)+\omega\cos(\sigma_1)\,\sqrt{x^2+z^2}\right)}{4\,c\,\pi\,\left(x^2+z^2\right)^{3/2}} \\ 0 \end{pmatrix}$$

where

$$\sigma_1 = \frac{\omega\,\left(c\,t-\sqrt{x^2+z^2}\right)}{c}$$

```
B_yt_xzE=int(ans(2),t)
```

```
B_yt_xzE =
```

$$\frac{a_0\,\mu\,q\,x\cos(\sigma_1)}{4\,\pi\,{\sigma_2}^{3/2}} - \frac{a_0\,\mu\,\omega\,q\,x\sin(\sigma_1)}{4\,c\,\pi\,\sigma_2}$$

where

$$\sigma_1 = \frac{\omega\,\left(c\,t-\sqrt{\sigma_2}\right)}{c}$$

$$\sigma_2 = x^2+z^2$$

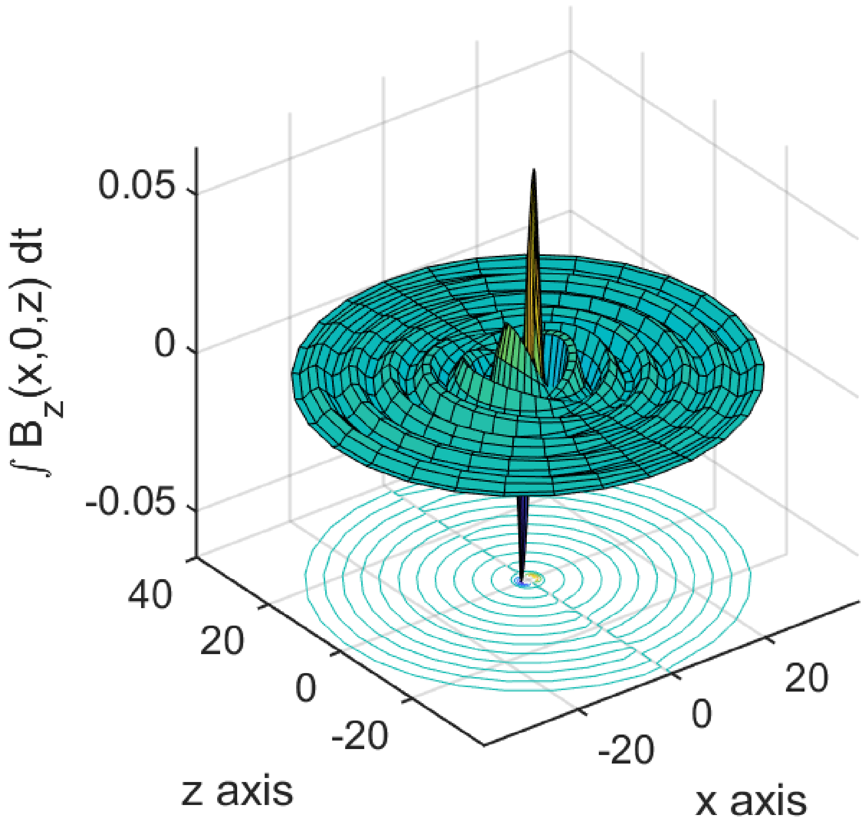
```
B_yt_xzEM=matlabFunction(sube(B_yt_xzE,Par))
```

```
B_yt_xzEM = function_handle with value:
    @(t,x,z)(x.*sin(t-sqrt(x.^2+z.^2)).*(-1.0./4.0))./(pi.*(x.^2+z.^2))+(x.*cos(t-sqrt(x.^2+z.^2)).*1.0./(x.^2+z.^2
```

```
N=30
```

```
N = 30
```

```
[R,The]=meshgrid(linspace(0,40,N),linspace(0,2*pi,N));
X=R.*sin(The);
Z=R.*cos(The);
subplot(1,1,1)
surfc(X,Z,B_yt_xzEM(0,X,Z))
axis square
xlabel('x axis')
ylabel('z axis')
zlabel('\int{ B_z(x,0,z) dt}')
```

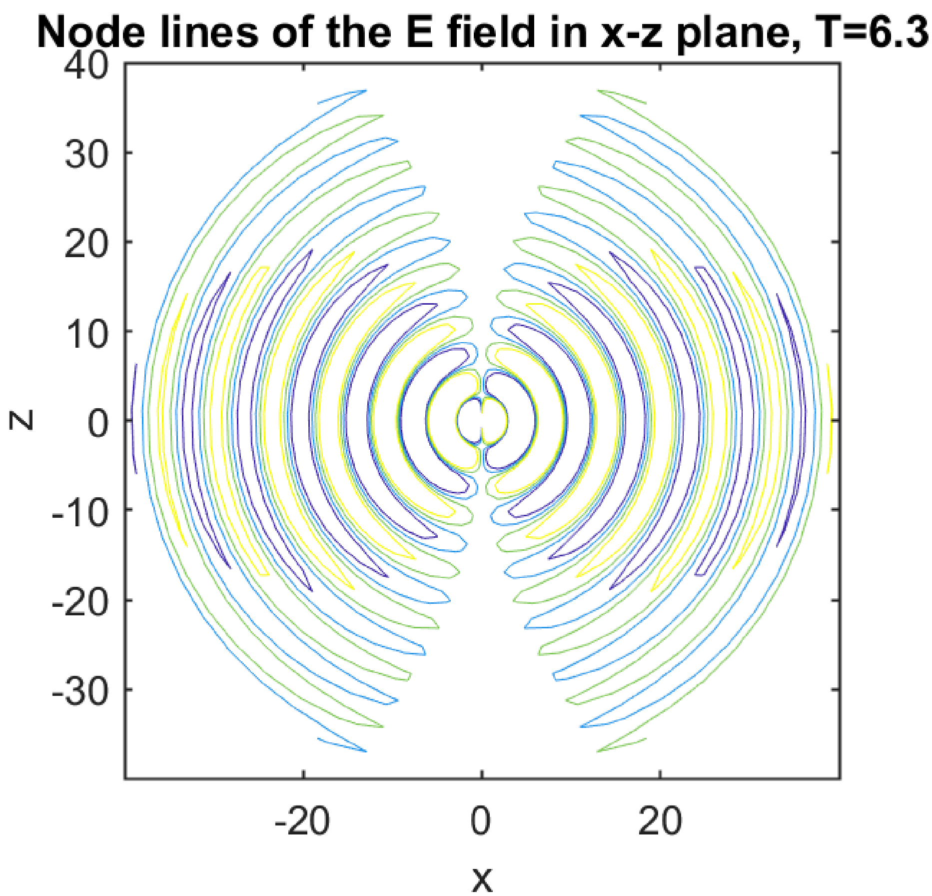


7 Animation of the field lines in the x-z plane (-> Hertz_Dipol.gif)

```
N=50;
[R,The]=meshgrid(linspace(0,40,N),linspace(0,2*pi,N));
X=R.*sin(The);
Z=R.*cos(The);

subplot(1,1,1)
clear iml
N=50;
T=linspace(0,2*pi,N);
n=1;
```

```
contour(X,Z,B_yt_xzEM(T(n),X,Z),0.002*linspace(-1,1,4))
axis square
xlabel('x')
ylabel('z')
title(sprintf('Node lines of the E field in x-z plane, T=%2f',T(n)))
im{n}=frame2im(getframe(gcf));
for n=1:size(T,2)
    contour(X,Z,B_yt_xzEM(T(n),X,Z),0.002*linspace(-1,1,4))
    axis square
    xlabel('x')
    ylabel('z')
    title(sprintf('Node lines of the E field in x-z plane, T=%2.1f',T(n)))
    im{n}=frame2im(getframe(gcf));
end
```




```
FiNa='Hertz_Dipol.gif';
n=1;
[A,map] = rgb2ind(im{n},8);
imwrite(A,map,FiNa,'gif','LoopCount',Inf,'DelayTime',0);
for n=2:size(T,2)
    [A,map]=rgb2ind(im{n},8);
    imwrite(A,map,FiNa,'gif','WriteMode','append','DelayTime',0);
end
```

Curl in generalized coordinates (function)

```
function Res=Curl3d(Ex,Ko,Me)
% Curl in generalized coordinates
% (Expression F,coordinates q, metric g)
```

$$\hat{e}_1\left(\vec{\nabla}\times\vec{F}\right)=\frac{1}{g_2g_3}\left(\frac{\partial}{\partial q^2}g_3F_3-\frac{\partial}{\partial q^3}g_2F_2\right)\quad(cyclical),\qquad g_i\equiv\left|\frac{\partial\vec{r}}{\partial q^i}\right|,\quad F_i\equiv\vec{F}\cdot\hat{e}_i$$

```
    Res=[...
        1/(Me(2)*Me(3))*(diff(Me(3)*Ex(3,1),Ko(2))-diff(Me(2)*Ex(2,1),Ko(3)));...
        1/(Me(3)*Me(1))*(diff(Me(1)*Ex(1,1),Ko(3))-diff(Me(3)*Ex(3,1),Ko(1)));...
        1/(Me(1)*Me(2))*(diff(Me(2)*Ex(2,1),Ko(1))-diff(Me(1)*Ex(1,1),Ko(2)))...
        ];
    Res=simplify(Res);
end
```