EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

MATHEMATISCH-
NATURWISSENSCHAFTLICHE FAKULTÄT
Fachbereich Physik

Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

Kurt Bräuer

6.5 Schwarzschild metric and black holes (Computational example)

The equivalence principle combines the continuity of the momentum gradient or the electromagnetic potential with the metric.

With the help of a suitable approach, the Schwarzschild metric can be determined.

From this follows that seen from the outside, in the field of a point-like gravitational source the clocks slow down and the space lines are stretched radially.

```
clear all
syms r theta phi a0 q mu omega ct e c r_s real
syms A(r) B(r)
Par=[a0==1 q==1 mu==1 omega==1 r_s==1]
```

Par = $\begin{pmatrix} a_0 = 1 & q = 1 & \mu = 1 & \omega = 1 & r_s = 1 \end{pmatrix}$

1 Schwarzschild metric

Approach: $(ds)^2 = A(r)(dr)^2 r^2 (d\vartheta)^2 + r^2 \sin^2 \vartheta (d\varphi)^2 - B(r)(dct)^2$

Metric: $G = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin\varphi & 0 \\ 0 & 0 & 0 & -B(r) \end{pmatrix}$

```
G=[[A(r) 0 0 0];[0 r^2 0 0];[0 0 r^2*sin(theta)^2 0];[0 0 0 -B(r)]]
```

G =

$$\begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & -B(r) \end{pmatrix}$$

```
G_i=G^-1
```

G_i =

$$\begin{pmatrix} \frac{1}{A(r)} & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2 \sin(\theta)^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{B(r)} \end{pmatrix}$$

2 Christoffel symbols

Definition: $\Gamma^{\lambda}_{\mu\nu} \equiv \vec{g}^{(4)\lambda} \cdot \frac{\partial \vec{g}^{(4)}_{\mu}}{\partial q^{\nu}}$

or $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}\left(\frac{\partial g_{\kappa\mu}}{\partial q^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial q^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial q^{\kappa}}\right)$

or for a diagonal metric: $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\lambda}\left(\frac{\partial g_{\lambda\mu}}{\partial q^{\nu}} + \frac{\partial g_{\lambda\nu}}{\partial q^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial q^{\lambda}}\right)$

```
Koord=[r,theta,phi,ct];
for lam=1:4
    dG{lam}=diff(G,Koord(lam));
end
Gamm=sym('g',[4,4,4]);
for lam=1:4, for mu=1:4, for nu=1:4
            Gamm(lam,mu,nu)=G_i(lam,lam)*(dG{nu}(lam,mu)+dG{mu}(lam,nu)-dG{lam}(mu,nu))/2;
        end, end, end
```

Output of non-disappearing symbols

```
for lam=1:4, for mu=1:4, for nu=1:4
            if ~isequal(Gamm(lam,mu,nu),sym(0))
                txt=sprintf('Gamma_%d%d^%d',mu,nu,lam);
                disp(str2sym(txt)==Gamm(lam,mu,nu))
            end
        end, end, end
```

$$\Gamma_{11} = \frac{\frac{\partial}{\partial r} A(r)}{2\, A(r)}$$

$$\Gamma_{22} = -\frac{r}{A(r)}$$

$$\Gamma_{33} = -\frac{r\sin(\theta)^2}{A(r)}$$

$$\Gamma_{44} = \frac{\frac{\partial}{\partial r}\ B(r)}{2\ A(r)}$$

$$\Gamma_{12}{}^{2} = \frac{1}{r}$$

$$\Gamma_{21}{}^{2} = \frac{1}{r}$$

$$\Gamma_{33}{}^{2} = -\cos(\theta)\sin(\theta)$$

$$\Gamma_{13}{}^{3} = \frac{1}{r}$$

$$\Gamma_{23}{}^{3} = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\Gamma_{31}{}^{3} = \frac{1}{r}$$

$$\Gamma_{32}{}^{3} = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\Gamma_{14}{}^{4} = \frac{\frac{\partial}{\partial r}\ B(r)}{2\ B(r)}$$

$$\Gamma_{41}{}^{4} = \frac{\frac{\partial}{\partial r}\ B(r)}{2\ B(r)}$$

3 Scharzschild metric B(r), A(r)

Equivalence principle: $\frac{\partial p_\nu}{\partial q^\mu} = e\frac{\partial A_\nu}{\partial q^\mu} = -m_0 g_{\mu\lambda}\Gamma^\lambda_{\kappa\nu}v^\kappa$

for $\nu = 4,\ \ \mu = 1,\ \ A_4 = \frac{e}{c}\frac{1}{r}\ \ \rightarrow\ \ e\frac{A_e}{r} = -m_0 G_{11}\Gamma^4_{1\kappa}v^\kappa$

with Schwarzschild radius r_s

```
syms m_0 r_s positive
v_v=[0;0;0;c]
```

v_v =

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix}$$

```
A_4(r)=e/(r*c)
```

A_4(r) =

$$\frac{e}{c\,r}$$

```
e*diff(A_4)==-m_0*G(1,1)*Gamm(1,:,4)*v_v
```

ans(r) =

$$-\frac{e^2}{c\,r^2} = -\frac{c\,m_0 \frac{\partial}{\partial r}\, B(r)}{2}$$

```
e1(r)=isolate(ans,diff(B,r))
```

e1(r) =

$$\frac{\partial}{\partial r}\, B(r) = \frac{2\,e^2}{c^2\,m_0\,r^2}$$

```
e2=r_s==sube(rhs(e1(r)),r==1)
```

e2 =

$$r_s = \frac{2\,e^2}{c^2\,m_0}$$

```
sube(e1,isolate(e2,e))
```

ans(r) =

$$\frac{\partial}{\partial r}\, B(r) = \frac{r_s}{r^2}$$

```
e3=B(r)==dsolve(ans,B(inf)==1)
```

e3 =

$$B(r) = 1 - \frac{r_s}{r}$$

Radial motion on light cone: $0 = (ds)^2 = A(r)^2(dr)^2 - B(r)^2(dct)^2 \;\; \rightarrow \;\; 1 = \frac{dr}{dct} = \frac{B(r)}{A(r)}$

```
A(r)*B(r)==1
```

ans = $A(r)\,B(r) = 1$

```
eq3=[e3,isolate(sube(ans,e3),A(r))]
```

eq3 =

$$\left(B(r) = 1 - \frac{r_s}{r} \quad A(r) = \frac{r}{r - r_s} \right)$$

4 Resting clock in the gravitational field

$(ds)^2 = A(r)(dr)^2 r^2 (d\vartheta)^2 + r^2 \sin^2 \vartheta (d\varphi)^2 - B(r)(dct)^2$

For object at rest: $(ds)^2 = G_{44}(dct)^2 = -(d\tau)^2$

Interpretation: A local time interval $d\tau$ at the event horizon appears infinitely long from the outside. Frequencies appear infinitely small from the outside.

```
syms dt
dtau=sqrt(sube(-G(4,4),eq3))*dt
```

dtau =

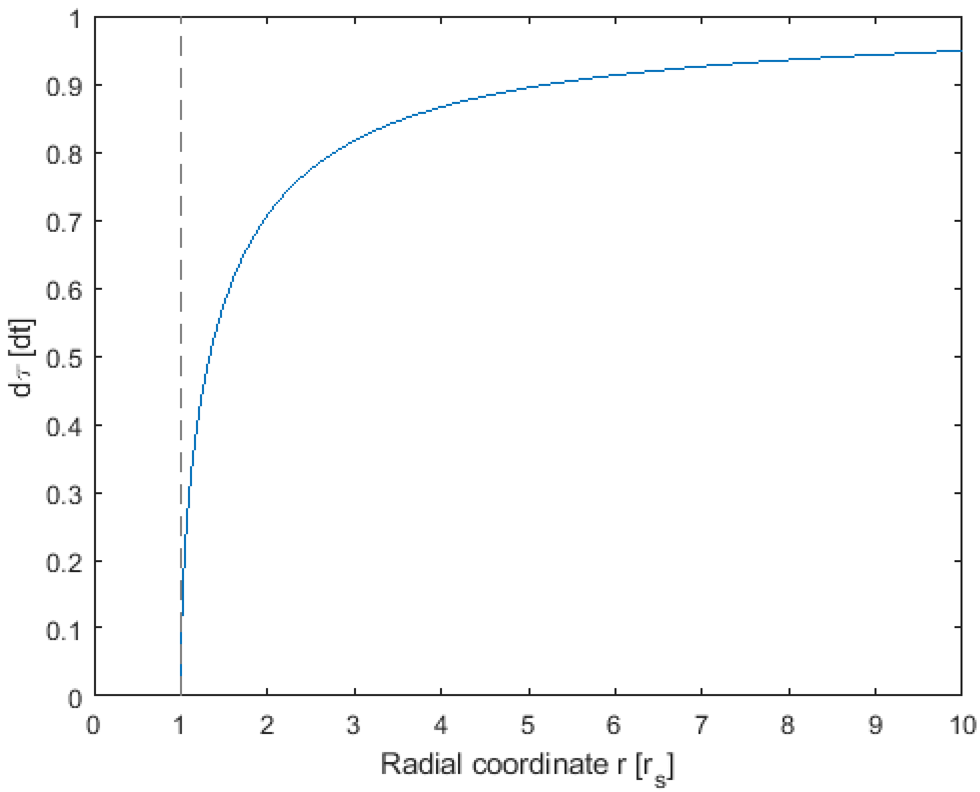
$$\mathrm{dt}\ \sqrt{1 - \frac{r_s}{r}}$$

```
sube(dtau,[dt==1,r_s==1])
```

ans =

$$\sqrt{1 - \frac{1}{r}}$$

```
fplot(ans,[0 10])
axis([0 10 0 1])
xlabel('Radial coordinate r [r_s]')
ylabel('d\tau [dt]')
```



5 Black hole

The Schwarzschild elongation can be represented by curvature of the space in an additional space dimension *H*. The degree of bulge in *H* direction is determined by the theorem of Pythagoras.

Condition for H: $dr^2 + dH^2 = g_{rr} dr^2$

therefore: $dH = \pm\sqrt{g_{rr} - 1} dr$

Hight: $H(r) = \pm \int_{H(r_s)}^{H(r)} dH$

Interpretation: viewed from the outside, a local length dr is stretched to $\sqrt{g_{rr}}\,dr$. The path to (or from) the event horizon ultimately becomes infinitely large. No light can cover this way. The event horizon therefore appears black.

A local observation (eg of an astronaut at the event horizon) is not affected.

```
syms dr dH
dr^2+dH^2==G(1,1)*dr^2
```

ans = $\mathrm{dH}^2 + \mathrm{dr}^2 = \mathrm{dr}^2\, A(r)$

```
sube(ans,eq3)
```

ans =

$$\mathrm{dH}^2 + \mathrm{dr}^2 = \frac{\mathrm{dr}^2\, r}{r - r_s}$$

```
isolate(ans,dH)
```

ans =

$$\mathrm{dH} = \frac{\mathrm{dr}\ \sqrt{r_s}}{\sqrt{r - r_s}}$$

```
sube(dH,ans)
```

ans =

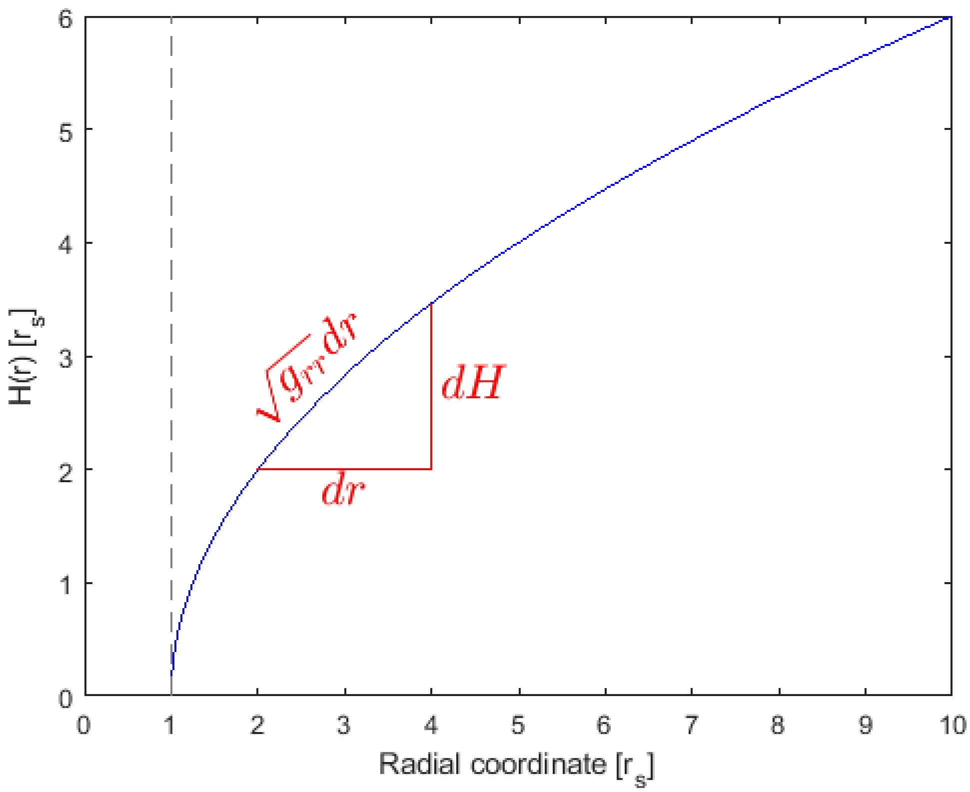
$$\frac{\mathrm{dr}\ \sqrt{r_s}}{\sqrt{r - r_s}}$$

```
int(ans/dr);
H=sube(ans,r_s==1)
```

H = $2\ \sqrt{r - 1}$

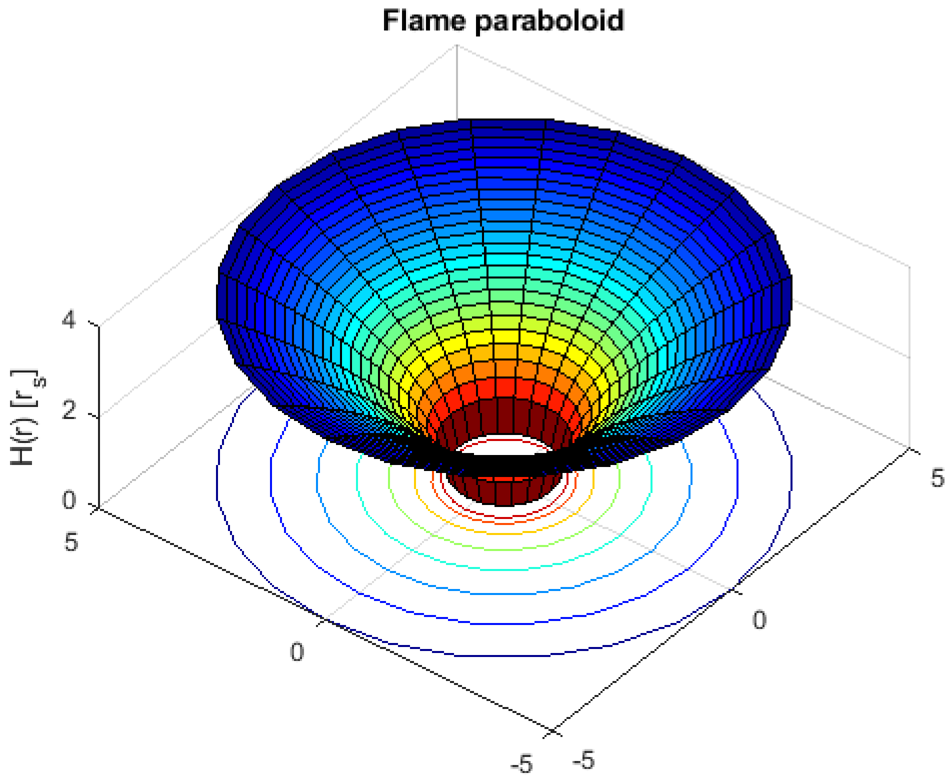
```
figure
set(gcf,'DefaultTextFontSize',18)
fplot(H,[0 10],'Color','b')
xlabel('Radial coordinate [r_s]')
hold on
Hn=matlabFunction(H);
plot([2 4 4],Hn([2 2 4]),'r')
ylabel('H(r) [r_s]')
text(1.9,2.5,'$\sqrt{g_{rr}} dr$','Interpreter','latex','Color','r',...
    'Rotation',40)
text(3,1.8,'$dr$','Color','r','Interpreter','latex', 'HorizontalAlignment','center')
text(4.1,mean(Hn([2 4])),'$dH$','Color','r','Interpreter','latex')

axis([0 10 0 6])
hold off
```



The so-called Flamm'sche paraboloid is created by rotation-symmetrical representation of $H(r)$

```
[r,phi]=meshgrid(linspace(1.01,5,25),linspace(0,2*pi,25));
matlabFunction(H);
surfc(r.*cos(phi),r.*sin(phi),ans(r))
axis equal
view([-50.30 30.00])
jet_=jet;
colormap(jet_(end:-1:1,:))
axis([-5 5 -5 5 0 4])
zlabel('H(r) [r_s]')
title('Flame paraboloid')
view([-51.72 38.54])
```



Flame paraboloid
H(r) [r_s]
4
2
0
5
0
-5
-5
0
5