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Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

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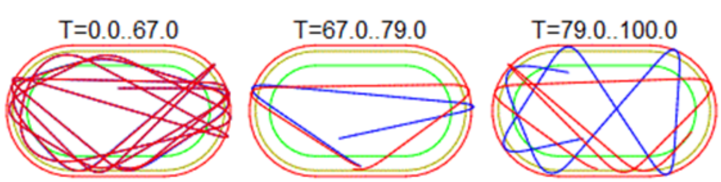
7.1 Chaos Theory

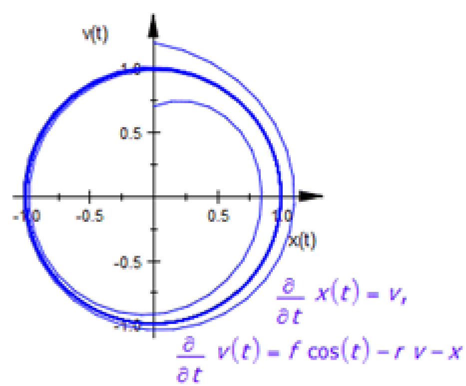
There are basically two types of chaotic systems

↙ ↘

Hamiltonian	Dissipative
Celestial bodies, billiards, ...	Oscillator with friction, Lorenz weather model,...

(7-1)





A simple Hamiltonian system is the free motion of a ball in a limited stadium-shaped area. At the boundary, the ball is elastically reflected. The smallest change in the start position leads to a completely different trajectory after about 10 reflections.

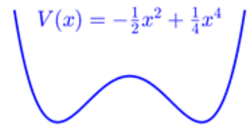
A simple dissipative system is a driven harmonic oscillator with friction. Regardless of the starting position, the movement ends on a phase space ellipse.

Such systems may result in attractors, fractals, and chaos.

Example: Duffing oscillator (dissipative)

This is a driven oscillator with friction and a two-trough potential.

Equation of motion: $\ddot{x}(t) + r \cdot \dot{x}(t) = -\frac{\partial}{\partial x} V(x)\Big|_{x=x(t)} + f \sin(\Omega \cdot t)$



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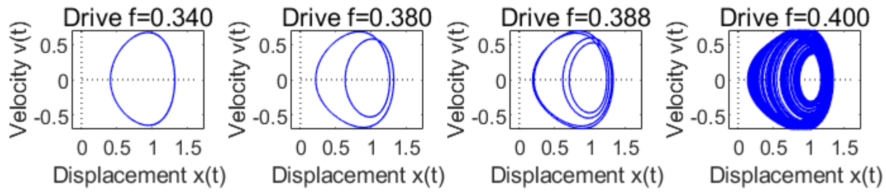


Depending on the drive amplitude *f*, the oscillator moves in the phase space in one cycle, or in 2 cycles, 4 cycles, 8 cycles, etcetera. At a certain drive amplitude, adjacent trajectories diverge exponentially, as in billiards above.

Deterministic chaos

Deterministic chaos is when these two conditions are met:

1. the time evolution of the state of a system from time t to time $t + \delta t$ is clearly determined
2. adjacent trajectories diverge exponentially

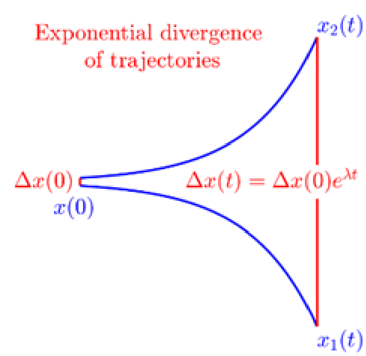
(7-3)

Exponential divergence refers to the temporal behavior of the distance between two adjacent trajectories

Exponential divergence: $\varepsilon(t) \equiv x_2(t) - x_1(t) = \varepsilon(0) e^{\lambda \cdot t}$

Liapunov exponent λ: $\lambda = \frac{1}{t} \ln \frac{\varepsilon(t)}{\varepsilon(0)}$

chaotic: $\lambda > 0$ | attractive: $\lambda < 0$



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Attractors in dynamic systems

The following definitions can be used to formulate the three properties of an attractor:

Flow map φ: $\varphi(\vec{x}(0), t) = \vec{x}(t)$

State space: $M = \{\varphi(\vec{x}(0), t)\}$ for all $\vec{x}(0), t$

Attractor: $\underbrace{A \subset M}_{\text{State-space}}$ is an attractor, if

1. A is invariant under the flow map: $\varphi(\vec{x}(0), t) \in A$ for all $\vec{x}(0) \in A, t \in \mathbb{R}^+$
2. A has an attractive environment U: for all $x(0) \in U$ applies $\lim_{t\to\infty} x(t) \in A$
3. and A is not splittable into sub spaces

(7-5)

Fractals

A fractal structure consists of itself. For example, in the Cantor set, each line consists of two lines and a space in between. This applies to every sublevel, so that something arises between lines with spatial dimension 1 and points with dimension 0. The dimension is broken or fractal.

Example: Cantor set

The dimension D can be expressed by the scaling s and the number N of the substructures, in the case of the Cantor set by 3 and 2:

$$N = s^D \rightarrow D = \frac{\ln N}{\ln s} = \frac{\ln 2}{\ln 3} = 0.6309$$

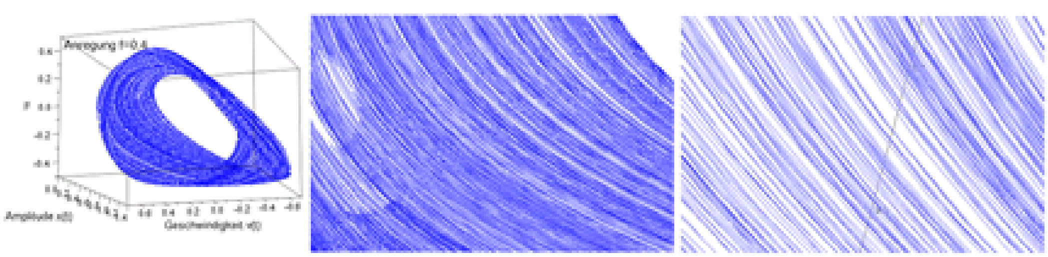
Strange attractor

A strange attractor has all of the following properties:

1. It is an attractor 2. It is chaotic 3. It has a fractal structure

(7-6)

In the Duffing Oscillator for a sufficiently large drive amplitude *f* an example can be found. The fractal structure can be recognized qualitatively by zooming into the attractor more and more.



Different magnifications of the duffing oscillator in the space of

position, speed and drive amplitude

Time evolution of the phase space

The temporal behavior of the phase space volume shows the essential difference between Hamiltonian and dissipative systems.

$$\text{Phase space:} \quad \vec{Q} = \begin{pmatrix} q \\ p \end{pmatrix}, \quad \dot{\vec{Q}} = \vec{F} = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$

$$\text{Volume element in phase space:} \quad dV = \det\underbrace{\left(\frac{\partial \vec{Q}}{\partial q}, \frac{\partial \vec{Q}}{\partial p}\right)}_{\text{Jacobian}} dqdp = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} dqdp = dqdp$$

(7-7)

Temporal change of the invariant volume element:

$$d\dot{V} = \left(\det\left(\frac{\partial \dot{\vec{Q}}}{\partial q}, \frac{\partial \vec{Q}}{\partial p}\right) + \det\left(\frac{\partial \vec{Q}}{\partial q}, \frac{\partial \dot{\vec{Q}}}{\partial p}\right)\right) dqdp = \left(\det\begin{pmatrix} \frac{\partial F_q}{\partial q} & 0 \\ \frac{\partial F_p}{\partial q} & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & \frac{\partial F_q}{\partial p} \\ 0 & \frac{\partial F_p}{\partial p} \end{pmatrix}\right) dV = \vec{\nabla} \cdot \vec{F} dV$$

	Duffing osc.	Hamilt. sys.
Divergence of F:	$\underbrace{\vec{\nabla} \cdot \vec{F} = -r}_{\vec{F}=\begin{pmatrix} p \\ -rp+\omega^2 q-\beta q^3+f\sin(\Omega t)\end{pmatrix}}$	$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial q} \cdot \underbrace{\dot{q}}_{\frac{\partial H}{\partial p}} + \frac{\partial}{\partial p} \cdot \underbrace{\dot{p}}_{-\frac{\partial H}{\partial q}} = 0$
invariant VE:	$d\dot{V} = -rdV, \quad dV = dV_0 e^{-rt}$	$d\dot{V} = 0$
hence:	$\lim_{t \to \infty} dV = 0$ für $r > 0$	$\lim_{t \to \infty} dV = dV$

(7-8)

Hamiltonian systems

Invariance of the volume

The invariance of the volume originates from Hamilton's equations of motion:

- V is conserved (*Liouville theorem*)!
- there are no attractors!
- chaotic behavior is possible

(7-9)

Integrability (no chaos)

Hamiltonian systems are called integrable if they can be fully determined by the calculation of one-dimensional integrals.

Under certain circumstances, multidimensional systems can be separated and therefore made integrable by choosing suitable generalized coordinates.

For example, the Kepler problem

Energy (constant)	Separation:
$E=\frac{1}{2m}\left(p_r^2+\frac{1}{r^2}p_\varphi^2\right)-\frac{c}{r}$	$\underbrace{p_\varphi^2}_{\neq f(r)}=\underbrace{r^2\left(2m\left(E+\frac{c}{r}\right)-p_r^2\right)}_{\neq f(\varphi)}=const$

(7-10)

Integration

Radial momentum	Integration	Inversion
$p_r(r)=\pm\sqrt{2m\left(E+\frac{c}{r}\right)-\frac{1}{r^2}p_\varphi^2}$ $=m\dot{r}$	$t=\int dt'=\int\frac{m}{p_r(r)}dr=t(r)$	$t(r)\ \rightarrow$ trajectory $\ r(t)$

(7-11)

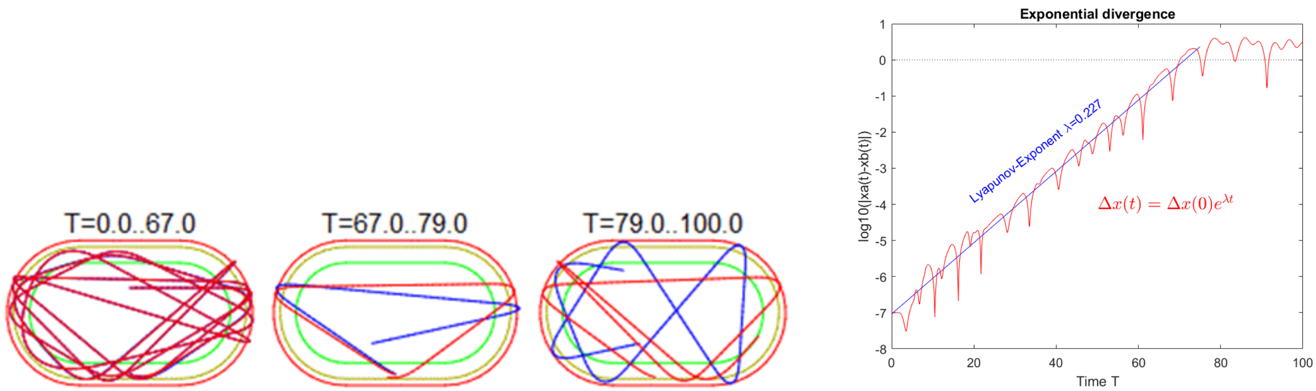
Other examples: Spherical pendulum, circular cone, heavy axis on an inclined plane, central force motion, heavy spinner, two-centre problem, swinging string of pearls.

Chaotic behavior in Hamiltonian systems

Non-separable systems may show an exponential divergence of neighboring trajectories. So, they are chaotic.

A sufficient condition for this behavior is the divergence of the classical perturbation theory. This explains for example the structure of the Saturn rings.

One can also numerically show the exponential divergence of adjacent trajectories, for example for the stadium problem (free motion with boundary conditions).



In the stadium problem, two closely adjacent trajectories diverge exponentially with time. On the left, the trajectories are shown in different time intervals. On the right, the logarithmic distance of the trajectories is shown as a function of time.

Further examples

Three-body problem (asteroids around sun, rings of Saturn, double pendulum, ...)