EBERHARD KARLS UNIVERSITÄT TÜBINGEN

MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT
Fachbereich Physik

Basic Physics Course with MATLAB's Symbolic Toolbox and Live Editor

Kurt Bräuer

7.3 Duffing oscillator (Computational example)

The Duffing oscillator is a one-dimensional driven oscillator with friction and a nonlinear potential. It is described by the

Equation of motion: $\ddot{x} + r\dot{x} = K + f\sin(\Omega t)$ where $K = -\frac{\partial}{\partial x}V$ and $V(r) = -\frac{1}{2}\omega^2 x^2 + \frac{1}{4}\beta x^4$

The equation of motion is solved numerically and the solutions are examined for attractive or chaotic behavior.

Considering:

- Phase space trajectories (x(t),v(t))
- Divergence and convergence of adjacent trajectories
- Poincare cuts

Depending on the drive amplitude f, the oscillator has different attractors (1-cycle, 2-cycle, 4-cycle, ...) and chaotic behavior.

```
clear all
syms omega Omega beta r x f x_0 v_0
Par=[r==0.25 omega==1 beta==1 Omega==1.4 x_0==0.001,v_0==0]
```

Par =

$$\left(r = \frac{1}{4} \quad \omega = 1 \quad \beta = 1 \quad \Omega = \frac{7}{5} \quad x_0 = \frac{1}{1000} \quad v_0 = 0\right)$$

```
F=[0.34 0.38 0.388 0.4]; %Drive amplitude
T=600;
N=4000;
Tn=linspace(0,T,N);
opt=odeset('RelTol',3e-14);
```

1 Potential of the Duffing oscillator

$$V = -\frac{1}{2}\omega^2 x^2 + \frac{1}{4}\beta x^4$$

```
V=-omega^2/2*x^2+beta*x^4/4
```

V =

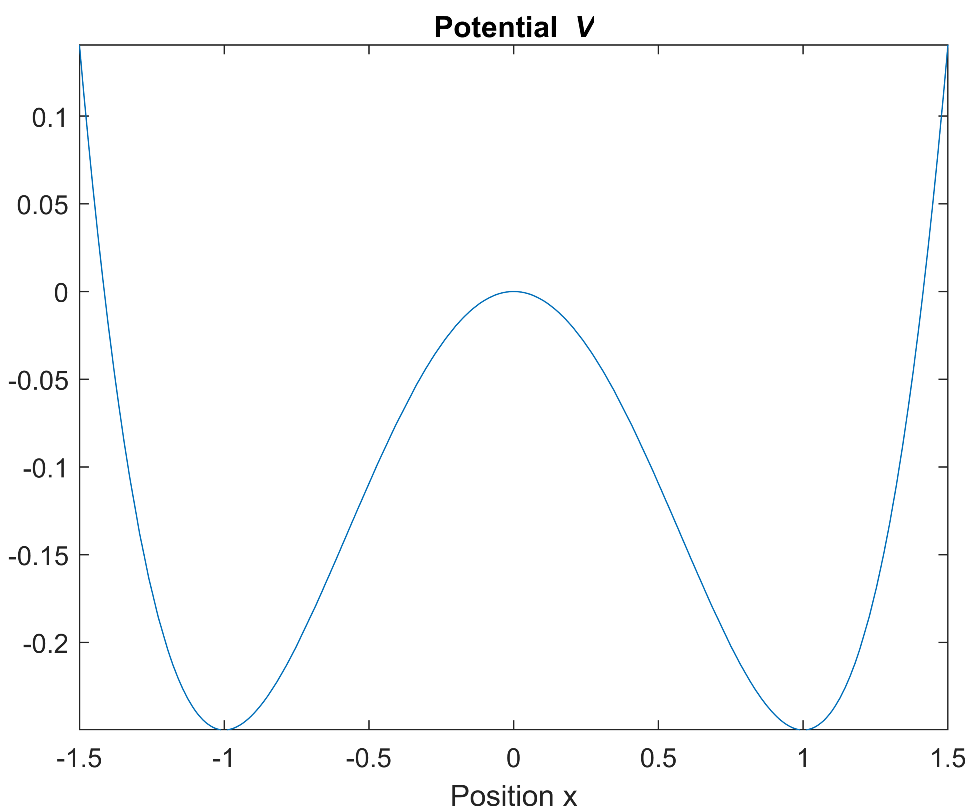
$$\frac{\beta\, x^4}{4} - \frac{\omega^2\, x^2}{2}$$

```
sube(V,Par)
```

ans =

$$\frac{x^4}{4} - \frac{x^2}{2}$$

```
fplot(ans,[-1.5 1.5])
xlabel('Position x')
title('Potential {\sl V}')
```



2 Equations of motion

$\ddot{x} + r\dot{x} = K + f\,\sin(\Omega t)$ mit $K = -\frac{\partial}{\partial x}V \rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = -rv + \omega^2 x - \beta x^3 + f\,\sin(\Omega t) \end{cases}$

```
syms x y(t) v(t)
K=-sube(diff(V,x),x==y(t));
syms x(t)
K=sube(K,y(t)==x(t))
```

K = $\omega^2\,x(t) - \beta\,x(t)^3$

```
diff(x(t),t,2)+r*diff(x(t),t)==K+f*sin(Omega*t)
```

ans =

$$\frac{\partial^2}{\partial t^2}\,x(t) + r\,\frac{\partial}{\partial t}\,x(t) = \omega^2\,x(t) - \beta\,x(t)^3 + f\,\sin(\Omega\,t)$$

```
isolate(ans,diff(x(t),t,2))
```

ans =

$$\frac{\partial^2}{\partial t^2}\,x(t) = -r\,\frac{\partial}{\partial t}\,x(t) + \omega^2\,x(t) - \beta\,x(t)^3 + f\,\sin(\Omega\,t)$$

```
DGL=[v(t),sube(rhs(ans),diff(x(t),t)==v(t))]
```

DGL = $\begin{pmatrix} v(t) & \omega^2\,x(t) - \beta\,x(t)^3 + f\,\sin(\Omega\,t) - r\,v(t) \end{pmatrix}$

3 Numerical solution and phase space representation

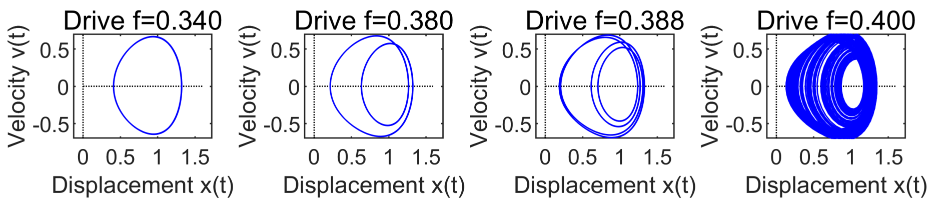
```
sube(DGL,Par)
```

ans =

$$\left(v(t) \quad -x(t)^3 + x(t) - \frac{v(t)}{4} + f\,\sin\left(\frac{7\,t}{5}\right)\right)$$

```
Fun=odeFunction(ans,[x(t) v(t)],f);
xv1=double(sube([x_0 v_0],Par));
xv2=double(sube([x_0+1e-7 v_0],Par));
l=0.8;
figure
for n=1:4
    [t_n,x1]=ode45(@(t,x)Fun(t,x,F(n)),Tn,xv1,opt);
    %subplot(2,2,n)
    subplot(4,4,n)
    plot([0 2*l],[0 0],':k')
    hold on
    plot([0 0],[-0.7 0.7],':k')
    plot(x1(1000:N,1),x1(1000:N,2),'b')
```

```
    text(0.8,l+.1,sprintf('Drive f=%3.3f',F(n)),...
        'HorizontalAlignment','center','Color','k')
    axis([0 1.5 -.7 .7])
    axis equal
    xlabel('Displacement x(t)')
    ylabel('Velocity v(t)')
    hold off
end
```



For the four drive amplitudes considered, the results are 1-cycle, 2-cycle, 4-cycle, and chaotic behavior with a strange attractor.

4 Convergence and divergence of the phase space trajectories

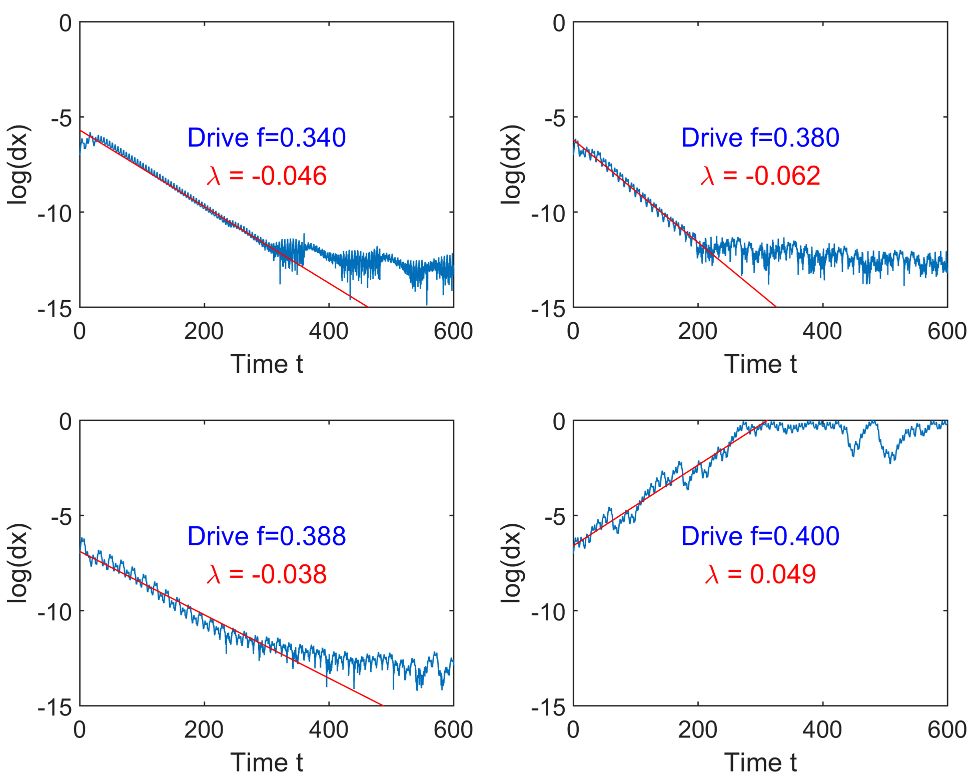
Two trajectories of the Duffing oscillator are considered, which initially have a small spatial distance of $x_2(0) - x_1(0) = 10^{-7}$ and the velocities $v_1(0) = v_2(0) = 0$.

The distance in phase space is $dx(t) = \left| \begin{pmatrix} x_2(r) - x_1(r) \\ v_2(r) - v_1(r) \end{pmatrix} \right|$

The Lyapunov exponent λ is defined by $dx(t) = dx(0)e^{\lambda t}$

```
n1=[1 1 1 1];
n2=[0.5*N 0.3*N 0.6*N N/2];
for n=1:4
```

```
    [t_n,x1]=ode45(@(t,x)Fun(t,x,F(n)),Tn,xv1,opt);
    [t_n,x2]=ode45(@(t,x)Fun(t,x,F(n)),Tn,xv2,opt);
    subplot(2,2,n)
    dx=log10((x1(:,1)-x2(:,1)).^2+(x1(:,2)-x2(:,2)).^2)/2;
    plot(t_n,dx)
    k=n1(n):n2(n);
    p=polyfit(t_n(k),dx(k),1);
    hold on
    plot([0 T],polyval(p,[0 T]),'r')
    text(300,-6,sprintf('Drive f=%3.3f',F(n)),...
        'HorizontalAlignment','center','Color','b')
    text(300,-8,['\lambda = ',sprintf('%3.3f',p(1)*log(10))],...
        'HorizontalAlignment','center','Color','r')
    axis([0 T -15 0])
    xlabel('Time t')
    ylabel('log(dx)')
    hold off
end
```



5 Poincare cuts

The displacement $x(t_n)$ of the oscillator at times $t_n = \frac{2\pi n}{\Omega}$ (n: cycles) is depicted.

This leads to a stroboscopic effect

```
N4=100;
```

```
dt=2*pi/Omega
```

dt =

$$\frac{2\,\pi}{\Omega}$$

```
dt=double(sube(dt,Par));
Tn4=0:dt:T;
for n=1:4
    [t_n,x1]=ode45(@(t,x)Fun(t,x,F(n)),Tn4,xv1,opt);
    subplot(2,2,n)
    plot(t_n,x1(:,1),'o','MarkerSize',2)
    xlabel('Time t')
    ylabel('x(t_n)')
    title(sprintf('Drive f=%3.3f',F(n)))
end
```