Boundary conditions of the hydrodynamic theory of electromagnetism

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Abstract

The complete set of boundary conditions for the hydrodynamic theory of polarizable and magnetizable media is obtained from the differential equations of the theory. The qualitative distinction of a different number of variables that exists between the dielectrics and conductors is stressed. The theory and the boundary conditions presented here are covariant to linear order in $v/c$.

Keywords: Hydrodynamic Maxwell equations; Boundary conditions; Covariance

1. Introduction

It was not until a few years ago that the Maxwell equations were incorporated into the hydrodynamic theory in a rigorous and consistent way, complete with dissipative terms and valid also in the nonlinear regime [1–4]. The resultant theory accounts for the low-frequency dynamics of macroscopic systems that are exposed to external fields, or contain electric charges and currents. It was helpful in understanding the curious spin-up behavior of ferrofluids [5], and has produced a number of interesting predictions [6,7].

The four following paragraphs are a quick summary, addressing those who are surprised by these statements, yet do not intend to read the Refs. [1–7]: The macroscopic Maxwell equations certainly belong to the best studied and verified differential equations in physics, especially in the optical range of frequency. Yet, they are usually taken as a set of stand-alone equations, decoupled from other macroscopic variables. So the evolution of the electromagnetic field is generally obtained with given and set values of density, temperature and local velocity, among others. In a few notable exceptions, these parameters are taken as functions of time. A rigorous and trustworthy theory, on the other hand, needs to consider these macroscopic, hydrodynamic quantities also as variables, and include their dynamics with that of the field. For instance, given the continuity equation for the momentum and its modification through the presence of an electromagnetic field, we will find an unambiguous answer to the notoriously difficult question of what is the ponderomotive force for a non-stationary, dissipative system of nonlinear constitutive relations [5,6]. This is a typical feed-back effect of how fields influence the hydrodynamic
variables. And it is simply neglected if the velocity is given as a parameter, even if it is a function of time.

Starting from the thermodynamic approach to electromagnetism as given in the justifiably famous Vol. 8 of Landau and Lifshitz [8], a theory that accounts for the dynamics of both the field and the hydrodynamic variables simultaneously is essentially what has been achieved in Refs. [1–7]. Most of the papers are confined to the true hydrodynamic regime of local equilibrium, though a first attempt of generalizing to the dispersive, optical frequency range has proven rather successful [6].

One striking result of this theory is the appearance of dissipative fields, \( H^D \) and \( E^D \) that are gradients of the original fields. They account for dissipative phenomena such as the restoration of equilibrium – even if the constitutive relations are nonlinear. (In contrast, the identification of the imaginary parts of \( \varepsilon \) and \( \mu \) with dissipation holds only for strictly linear constitutive relations.) Despite the \( k \)-dependence, or the appearance of spatial dispersion, \( H^D \) and \( E^D \) include temporal dispersion. (In fact, even in the linear case, these dissipative fields are more general that the accounts of temporal dispersion. The difference is large in select systems, especially dielectric ferrofluids [7].)

The higher-order gradient terms in the dissipative fields necessitate additional boundary conditions for the Maxwell equations, although no additional variables are being considered. This is the principle difference to the many works on the subject of “additional boundary conditions”, where either higher-frequency dynamic variables such as the excitonic degrees of freedom, or low-frequency, surface variables are being considered [9,10].

One of the main foundations of this new, hydrodynamic theory of electro-magnetism is a pragmatic combination of the Galilean and the Lorentz transformation to first order in \( v/c \): The former is employed for all hydrodynamic variables, including the temporal and spatial derivative, and the latter used for the field variables. As a result, the equations of motion are not covariant. This is cause for worry, because although the difference between the Galilean and the first-order Lorentz transformation is known to be irrelevant in usual hydrodynamic theories, circumstances are not as clear-cut when electromagnetic fields are involved.

This is the reason a fully relativistic version of the same theory was derived [11,12] employing only the Lorentz transformation, and demonstrating unambiguously the consistency of the hydrodynamic theory of electromagnetism. While this is clearly the correct theory to apply for relativistic systems such as quickly spinning astrophysical objects, it falls short when used to decide whether the original theory is indeed the appropriate and rigorously valid one for non-relativistic systems. For this purpose, one needs to go one step further, derive the complete set of boundary conditions, and thoroughly check whether any term not contained in the original theory could conceivably be important. The first part of this work is the content of the present paper.

The hydrodynamic theory and the boundary conditions, covariant to first order in \( v/c \), are presented here. We distinguish clearly between dielectrics and conductors, firstly because the theory for conductors has less independent variables, and hence also less boundary conditions, and secondly because the transition between the two theories is subtle and prone to errors.

The boundary conditions are derived from the bulk equations themselves, with the help of irreversible thermodynamics. This deviates from the strictly mathematical concept of differential equations, where boundary conditions are extrinsic information, used to select a special solution out of the manifold satisfying the equation.

The most well-known example of the approach in physics is the derivation of boundary conditions from the Maxwell equations, yielding results such as the continuity of the normal component of the magnetic field \( B \) [8]. There are many more recent examples: boundary conditions have been derived, especially, for broken symmetry systems, (such as liquid crystals [13], superfluid \(^4\)He [14,15] and \(^3\)He [16–18], including the \( A \rightarrow B \) transition [19–22],) but also for the more mundane shear flows of isotropic liquids [23].

The boundary conditions to be derived below are being applied to understand the viscosity of ferrofluids, and to dynamo theory. These results will be presented elsewhere. (In fact, one or two of the more unconventional boundary conditions have already been used [5,7], though without proper derivation or presentation.)
The paper has four more sections, the first two to present the hydrodynamic theory, for dielectrics and conductors, respectively, and the third to set up the boundary conditions. A fourth section considers two specific situations to clarify the physics contained in some of the boundary conditions.

2. The hydrodynamic theory for dielectrics

2.1. Lorentz transformation and thermodynamics

The starting point of every hydrodynamic theory is the thermodynamic theory. The energy density $\varepsilon_{\text{tot}}$ is a function of all the other conserved quantities, the entropy density $s$ and the field variables $B$ and $D$. For a two-component liquid we have, in the rest frame,

$$d\varepsilon_{\text{tot}} = d\varepsilon + c^2 d\rho$$

$$= T ds + (\mu + c^2) d\rho + \mu_c d\rho_c + H \cdot dB + E \cdot DD,$$

where the constraints

$$V \cdot B = 0, \quad V \cdot D = \rho_{el}$$

are satisfied in equilibrium. ($\rho_{el}$ is the charge density.) Note that $\varepsilon_{\text{tot}}$ is the total energy including the rest mass. The mass densities, $\rho$ and $\rho_c$, are connected to the particle numbers $n_1$ and $n_2$ by

$$\rho = m_1 n_1 + m_2 n_2, \quad \rho_c = m_2 n_2,$$

where $m_1$ and $m_2$ denote the respective masses. The choice of $\mu + c^2$ as the conjugate variable to $\rho$ renders the expansion in the small parameter $v/\rho c^2$ simple. (At most densities, the rest mass is certainly by far the dominating contribution.)

The equilibrium fluxes of energy and momentum in the rest frame are [5]

$$Q = cE \times H,$$

$$\Pi_{ij} = (Ts + \mu \rho + \mu_c \rho_c + E \cdot D + H \cdot B - v)\delta_{ij}$$

$$- \frac{1}{2}[H_i B_j + E_i D_j + (i \leftrightarrow j)]. \quad (5)$$

These variables and their fluxes constitute the energy – momentum 4-tensor $\Pi_{\mu \nu}$.

$$\Pi^{00} = \varepsilon_{\text{tot}}, \quad \Pi^{0k} = \Pi^{k0} = Q_k/c = c\varepsilon_{\text{tot}},$$

$$\Pi^{ik} = \Pi_{ik} = \Pi^{ki}. \quad (6)$$

The local conservation laws ensure that $\Pi_{\mu \nu}$ satisfies

$$\partial_{\nu} \Pi_{\mu \nu} = 0 \quad (7)$$

and is symmetric. (The greek indices go from 0 to 3, the latin ones from 1 to 3.)

The Lorentz transformation will now yield these thermodynamic expressions for an arbitrary inertial frame. Denoting the rest frame quantities with the superscript 0, the Lorentz transformation

$$\Pi_{\mu \nu} = \Lambda^\mu_\alpha [\Pi^{\alpha \beta}]^0 \Lambda_\beta^\nu \quad (8)$$

for $\nu = ve_x$ employs with the matrix

$$\Lambda_\nu^x = \begin{pmatrix} \gamma & \gamma \beta & 0 & 0 \\ \gamma \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (9)$$

where

$$\beta = v/c, \quad \gamma = 1/\sqrt{1 - \beta^2}. \quad (10)$$

Up to order $v^2/c^2$, we have

$$\varepsilon_{\text{tot}} = (1 + v^2/c^2)\varepsilon_{\text{tot},0} + 2v[1 + v^2/c^2]g_{\text{tot},0}$$

$$+ v_\mu \Pi_{\mu \nu}^{0} / c^{2}, \quad (11)$$

$$c_{\text{g}i}^{\text{tot}} = Q_i / c = v_i/c(1 + v^2/c^2)\varepsilon_{\text{tot},0}$$

$$+ [1 + v^2/(2c^2)]c_{\text{g}i}^{\text{tot},0} + 3v_i c_{\text{g}i}^{\text{tot},0} v_i/(2c^2)$$

$$+ [1 + v^2/(2c^2)]v_i \Pi_{\mu \nu}^{0}/c + v_i v_\mu \Pi_{\mu \nu}^{0}/c v_\nu/(2c^2), \quad (12)$$

$$\Pi_{ik} = v_i v_k \varepsilon_{\text{tot},0}/c^2 + [1 + v^2/(2c^2)]v_\mu c_{\text{g}i}^{\text{tot},0}$$

$$+ (1 + v^2/(2c^2)]v_\mu c_{\text{g}i}^{\text{tot},0} v_i/(v c^2)$$

$$+ \Pi_{ik}^{0} + \Pi_{ik}^{\text{tot},0}/(2c^2) + \Pi_{ik}^{0}/(2c^2). \quad (13)$$
As $\varepsilon \ll \rho c^2$, we shall only include terms of linear order in $v/c$, except for terms connected to the rest mass, $\rho c^2$, where quadratic orders will be included. (In the energy flux, this leads to a third-order term.) Then the lab-frame expressions for the energy, momentum and the fluxes are

$$g^{\text{tot}} = (Ts + \mu \rho + \mu_c \rho_c + \rho c^2)w/c^2 + E \times H/c,$$  

$$Q = c^2 g^{\text{tot}} + v_\parallel g,$$  

\[\Pi_{ij} = (Ts + \mu \rho + \mu_c \rho_c + v_\parallel g + E \cdot D + H \cdot B - \varepsilon)\delta_{ij} + \frac{1}{2}[g_\parallel v_j - E_i D_j - H_i B_j + (i \leftrightarrow j)],\]

where

$$g = g^{\text{tot}} - D \times B/c.$$

The other thermodynamic variables also need to be transformed. First, the four fields:

$$B^0 = B - v \times E/c,$$  

$$D^0 = D + v \times H/c,$$  

$$H^0 = H - v \times D/c,$$  

$$E^0 = E + v \times B/c.$$  

Both chemical potentials, $\mu + c^2$ and $\mu_c$, obey analogous formulas:

$$\mu + c^2 = [1 - v^2/(2c^2)](\mu^0 + c^2),$$  

$$\mu_c = [1 - v^2/(2c^2)]\mu_c^0.$$  

However, because of $c^2$, only $\mu$ is altered

$$\mu = \mu^0 - v^2/2,$$  

while $\mu_c = \mu_c^0$ remains invariant in linear order of $v/c$. The quantities $s, \rho_c$ and $T$ are also invariant to this order, $s = s^0, \rho_c = \rho_c^0$ and $T = T^0$. In the combination $\rho c^2$, we have

$$\rho = [1 + v^2/(2c^2)]\rho^0,$$  

otherwise, it is $\rho = \rho^0$.

Because $\varepsilon \ll \rho c^2$, the nonrelativistic expression for $g^{\text{tot}}$ is

$$g^{\text{tot}} = \rho v + E \times H/c.$$  

Putting all this together, the $\Pi^{00}$-component yields

$$\delta \varepsilon = \delta g^{\text{tot}} - c^2 \delta \rho = T ds + \mu \delta \rho + \mu_c \delta \rho_c + v \cdot \delta g + H \cdot \delta B + E \cdot \delta D.$$  

The isotropy of the system, i.e. the invariance of the energy density under an infinitesimal rotation implies:

$$v \times g + H \times B + E \times D = 0.$$  

2.2. The equilibrium conditions

Maximizing the integrated entropy $\int d^3 r s$ under the constrains of various conservation laws and Eq. (2), the resulting Euler–Lagrange equations, or the equilibrium conditions are [11,12]

$$\partial_t T = 0, \quad \partial_t \mu = 0, \quad \partial_t \mu_c = 0,$$  

$$\partial_\parallel H^0 = 0, \quad \partial_\parallel E^0 = 0, \quad v_{ij} = 0, \quad \partial \mu + \partial_\parallel v + \mu_\parallel \partial_\parallel v/c^2 = 0, \quad \partial^0 T = \frac{1}{\mu + c^2} T V^0 \mu = 0, \quad \partial^0 \mu = \frac{1}{\mu + c^2} V^0 \mu = 0.$$  

Note the preponderance of rest frame quantities. Except for the chemical potential, all spatial derivatives are small. If the charges are eventually able to move around, the total entropy may be further maximized if

$$E^0 = 0.$$  

2.3. The hydrodynamic equations

In local equilibrium, the conditions of Eq. (27) are not met, and the left-hand sides (that we shall call thermodynamic forces) are not zero. This leads to dissipative terms (denoted below by the superscript $D$) and entropy production as functions of the thermodynamic forces, which parameterize the
deviation from global equilibrium. The variables, of course, still obey continuity equations,

\[ 0 = \partial_t (\mathbf{B} + \mathbf{B}^D) + c\nabla \times (\mathbf{E} + \mathbf{E}^D), \quad (30) \]

\[ 0 = \partial_t (\mathbf{D} + \mathbf{D}^D) + f_i^D + (\rho c + \rho_c^D) \mathbf{v} - c\nabla \times (\mathbf{H} + \mathbf{H}^D), \quad (31) \]

\[ 0 = \partial_t (\rho + \rho^D) + \nabla \cdot (\rho \mathbf{v} - \mathbf{f}^D), \quad (32) \]

\[ 0 = \partial_t (\rho c + \rho_c^D) + \nabla \cdot (\rho_c \mathbf{v} - \mathbf{f}_c^D), \quad (33) \]

\[ R/T = \partial_t (s + s^D) + \nabla \cdot (s \mathbf{v} - \mathbf{f}^D), \quad (34) \]

\[ 0 = \partial_t (s^\text{tot} + s_i^\text{tot,D}) + \nabla \cdot (\Pi_{ij} - \Pi_{ij}^D), \quad (35) \]

\[ 0 = \partial_t (\varepsilon^\text{tot} + \varepsilon^D) + \nabla \cdot (\mathbf{Q} + \mathbf{Q}^D) \]

with

\[ \nabla \cdot (\mathbf{B} + \mathbf{B}^D) = 0, \quad (37) \]

\[ \nabla \cdot (\mathbf{D} + \mathbf{D}^D) = \rho c + \rho_c^D. \quad (38) \]

We may subtract the equation of motion for mass from that of the total energy \( \varepsilon^\text{tot} \), to arrive at the continuity equation for the nonrelativistic form of energy conservation, more usual in hydrodynamic theories,

\[ 0 = \partial_t (\varepsilon + \varepsilon^D - \rho^D c^2) + \nabla \cdot (\mathbf{Q} + \mathbf{Q}^D - \rho c^2 \mathbf{v} + c^2 \mathbf{f}^D). \quad (39) \]

The covariance of Eqs. (30)–(38) has a number of consequences:

- The equilibrium contributions of energy, momentum and their fluxes constitute the equilibrium energy–momentum 4-tensor \( \Pi^\text{eq} \); the same applies to the nonequilibrium contributions, \( \varepsilon^D, \mathbf{Q}^D, s_i^\text{tot,D} \) and \( \Pi_{ij}^D \) constitute the nonequilibrium 4-tensor \( \Pi_{ij}^D, \text{eq} \).

- Analogously, all fields, the equilibrium and nonequilibrium ones, \( \mathbf{E}, \mathbf{B}, (\mathbf{E}^D, \mathbf{B}^D), (\mathbf{D}, \mathbf{H}) \) and \( (\mathbf{D}^D, \mathbf{H}^D) \) constitute 4-tensors of the form

\[
(E, B)^{\text{eq}} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix}. \quad (40)
\]

- The equilibrium quantities \( (s, sv/c, \rho, \rho v/c, \rho c, \rho_c v/c) \) and \( (\rho_c, \rho, \rho v/c) \) are 4-vectors. The same applies to the nonequilibrium quantities \( (s^D, sv/c, \rho^D, \rho v/c, \rho_c^D, \rho_c v/c) \) and \( (\rho, \rho_c, \rho, \rho v/c) \).

The dissipative contributions of the variables \( s^D, \rho^D, \rho_c^D, g_i^\text{tot,D}, \mathbf{B}^D, \mathbf{D}^D \) and \( \rho_c^D \) vanish in the local rest frame, \( v = 0 \), because this is where the nonrelativistic and relativistic physics overlap. Here, according to the concept of local equilibrium, the variables only contain equilibrium information. As a result, their form in an arbitrary frame is given as

\[ \mathbf{B}^D = v \times \mathbf{E}^D/c, \quad \mathbf{D}^D = -v \times \mathbf{H}^D/c, \quad (41) \]

\[ \rho^D = v \cdot \mathbf{f}^D/c^2, \quad s^D = -v \cdot \mathbf{f}^D/c^2, \quad \rho_c^D = -v \cdot \mathbf{f}_c^D/c^2, \quad (42) \]

\[ g_i^\text{tot,D} = -\Pi_{ij}^D v_j/c^2 \quad (43) \]

and because of the symmetry of the energy–momentum 4-tensor,

\[ \mathbf{Q}^D = c^2 g_i^\text{tot,D} = -v \Pi_{ij}^D \quad (45) \]

with

\[ \varepsilon^D = O(v^2/c^2) \quad (46) \]

Yet another order higher.

To actually obtain the dissipative currents, we substitute \( \partial_t \varepsilon^\text{tot} \) in Eq. (36) for the expressions of Eq. (25),

\[ T \partial_s - (\mu + c^2 \varepsilon) \partial_c + \varepsilon \partial_c v + v \cdot \partial_c \mathbf{g} + \mathbf{H} \cdot \partial_c \mathbf{B} + \mathbf{E} \cdot \partial_c \mathbf{D} + \partial_c v \mathbf{D} = -\nabla \cdot (\mathbf{Q} + \mathbf{Q}^D), \quad (47) \]

insert the appropriate equations of motion, Eqs. (30)–(35), excluding terms of order higher than \( v/c \), and arrive at two expressions, the energy flux \( \mathbf{Q}^D \) and the entropy production \( R \). The energy flux is

\[ \mathbf{Q}^D = -\left( \mu + c^2 \right) g_i + \mu_c j_i - T f_i^D \]

\[ + c [\mathbf{E}^D \times \mathbf{H}^0 + \mathbf{E}^0 \times \mathbf{H}^D]_x - v_i \Pi_{ij}^D \quad (48) \]

which in conjunction with Eq. (45) leads to

\[ (\mu + c^2) f_i^D = -\mu_c j_i^D - T f_i^D \]

\[ + c [\mathbf{E}^D \times \mathbf{H}^0 + \mathbf{E}^0 \times \mathbf{H}^D]_x. \quad (49) \]
that the displacement current \( j_0^D \sim c^{-2} \) is relativistically small, and it is well justified to ignore it in non-relativistic theories.

The entropy production is

\[
R = f^D \cdot V^0 T + \Pi_{ij}^D \nu_{ij} + j^D \cdot V^0 \mu_e + \mathbf{j}^D \cdot \mathbf{V}^0 \mu - \mathbf{H}^D \cdot c (\mathbf{V}^0 \times \mathbf{E}^0) 
\]

Inserting Eq. (49), the number of independent thermodynamic forces is reduced by one,

\[
R = f^D \cdot \left( V^0 T - \frac{1}{\mu + c^2} T V^0 \mu \right) 
+ \Pi_{ij}^D \nu_{ij} + j^D \cdot \left( V^0 \mu_e - \frac{1}{\mu + c^2} \mu_e V^0 \mu \right) 
+ \mathbf{j}^D \cdot E^D \cdot c \left( V^0 \times H^0 + \frac{1}{\mu + c^2} H^0 \times V^0 \mu \right) 
- \mathbf{H}^D \cdot c \left( V^0 \times E^0 + \frac{1}{\mu + c^2} E^0 \times V^0 \mu \right).
\]

It is instructive to compare this equation with Eq. (27). Clearly, the thermodynamic forces there appear here again. If they vanish, we have equilibrium, and the entropy production \( R \) is zero. If they do not, the leading terms in \( R \), being a positive-definite function, must be quadratic in these forces. In this order, the dissipative currents, \( f^D, \Pi_{ij}^D, j^D, E^D, H^0, \mathbf{j}_0^D \), are proportional to the forces, with the coefficients obeying the Onsager symmetry relations.

3. The hydrodynamic theory for conductors

If the electrical conductivity is large enough, such that the displacement current \( \partial_t \mathbf{D} \) in

\[
0 = \dot{\mathbf{D}} + \mathbf{j}_0^D + \rho \mathbf{v} - c \nabla \times (\mathbf{H} + \mathbf{H}^0) \tag{52}
\]

is negligible with respect to \( \mathbf{j}_0^D = \sigma \mathbf{E}^0 \), it may then be eliminated. This is the quasi-stationary approximation [8], always justified in the low-frequency limit: Consider (for \( \mathbf{v} \equiv 0 \))

\[
|\dot{\mathbf{D}}| \approx \omega \varepsilon E \ll j_0^D = \sigma E, \tag{53}
\]

(where \( \varepsilon \) is the dielectric constant,) and find

\[
\omega \varepsilon / \sigma \ll 1. \tag{54}
\]

We shall in this work refer to a system as a conductor if the quasi-stationary approximation is valid. (If not, the displacement current \( \partial_t \mathbf{D} \) needs to be included and, despite some residual conductivity, the equations of the last section apply.)

In the Heaviside – Lorentz system of units, the conductivity of metals is of the order of \( \sigma = 10^{18}/s \), eight to ten orders of magnitude above the frequencies at which local equilibrium reigns and the hydrodynamic theory is valid. So the quasi-stationary approximations always apply in these systems, which can therefore be considered as conductors without any qualification. If the conductivity is below \( 10^{10}/s \), circumstances are more complicated and depend on the given frequency.

The elimination of the displacement current \( \partial_t \mathbf{D} \) is a qualitatively important step, as this is equivalent to the statement that \( \mathbf{D} \) is no longer an independent variable of the hydrodynamic theory. Indeed, the relaxation time of the electric field is \( \tau = \varepsilon / \sigma \), as

\[
|\dot{\mathbf{E}}| \approx E / \tau \approx \dot{\mathbf{D}} / \varepsilon \approx \sigma \mathbf{E} / \varepsilon \tag{55}
\]

and the electric field has ample time to relax for frequencies \( \omega \tau \ll 1 \). So, taking \( \mathbf{E}^0 \) as zero in equilibrium, the rest frame thermodynamics is now,

\[
d \varepsilon^{\text{emf}} = T \, ds + (\mu + c^2) \, d\rho + \mu \varepsilon \, d\rho_e + \mathbf{H} \cdot d\mathbf{B}. \tag{56}
\]

Note that the electric field need not vanish in the lab-frame, \( \mathbf{v} \neq 0 \), though of course only via the Lorentz transformation

\[
\mathbf{E} = - \mathbf{v} \times \mathbf{B}/c, \quad \mathbf{D} = - \mathbf{v} \times \mathbf{H}/c, \tag{57}
\]

without changing the fact that they are not independent. Note also that we have only implemented the fact that the equilibrium electric field vanishes. Off equilibrium, it does not, and as we know may drive a current through a wire. It remains dependent, however, and is in this case given by

\[
c \nabla \times (\mathbf{H} + \mathbf{H}^0) = \sigma \mathbf{E}. \tag{58}
\]
There are two ways to arrive at the hydrodynamic theory for conductors, one may either set to zero all terms $N D^0$ and $E^0$ in the previous theory, or one may start from scratch with a reduced set of variables. Both methods lead to the same results, which are presented below.

The basic thermodynamic identity for an arbitrary inertial system is,

$$d\varepsilon = dv^{\text{tot}} - c^2 d\rho = T \, ds + \mu \, d\rho + \mu_c \, d\rho_c + v \cdot dg^{\text{tot}} + H \cdot dB,$$

where

$$g^{\text{tot}} = (Ts + \mu \rho + \mu_c \rho_c + \rho c^2) \frac{v}{c^2} - \frac{1}{c}(v \times B) \times H$$

In equilibrium, the magnetic field satisfies

$$V \cdot B = 0.$$  \hspace{1cm} (61)

To linear order in $v/c$, the fields

$$H^0 = H, \quad B^0 = B$$  \hspace{1cm} (62)

are Lorentz invariant.

The change in the equilibrium conditions are in the Euler–Lagrange equations for the field. Only

$$\partial_t H = 0, \quad V^0 \times H + \frac{1}{\mu + c^2} H \times V^0 \mu = 0,$$

remain, with $E^0 \equiv 0$ implied. The rotational identity Eq. (26) reduces to

$$H \times B = 0.$$  \hspace{1cm} (64)

The hydrodynamic equations are

$$0 = \partial_t (B + B^D) + c V \times (-v \times B/c + E^D),$$

$$0 = \partial_t (\rho + \rho^D) + V \cdot (\rho v - j^D),$$

$$0 = \partial_t (\rho_c + \rho_c^D) + V \cdot (\rho_c v - j_c^D),$$

$$R \frac{d}{dt} = \partial_t (s + s^D) + V \cdot (sv - f^D),$$

$$0 = \partial_t(g^{\text{tot}} + g^{\text{tot},D}) + V(j + j^{\text{tot},D}),$$

$$0 = \partial_t(\xi + \xi^D - \rho^D c^2) + V \cdot (Q + Q^D - \rho c^2 v + j^D)$$

with

$$0 = V \cdot (B + B^D), \quad B^D = v \times E^D/c,$$  \hspace{1cm} (71)

$$Q = (Ts + \mu \rho + \mu_c \rho_c + \rho c^2 + v \cdot g)v - (v \times B) \times H,$$  \hspace{1cm} (72)

$$g = g^{\text{tot}} + (v \times H) \times B/c^2 = \rho v,$$  \hspace{1cm} (73)

$$\Pi_{ij} = [Ts + \mu \rho + \mu_c \rho_c + v \cdot g + H \cdot B - \partial_s] \delta_{ij}$$

$$+ \frac{1}{4}[g_{ij} v_j - H_i B_j + (i \leftrightarrow j)],$$  \hspace{1cm} (74)

$$(\mu + c^2) j_i^D = -T f_i^D - \mu_c j_{c,i}^D + c[E^D \times H],$$  \hspace{1cm} (75)

$$R = f^D \cdot \left(V^0 T - \frac{1}{\mu + c^2} T V^0 \mu \right)$$

$$+ \Pi_{ij} v_{ij} + j_{c} \cdot \left(V^0 \mu_c - \frac{1}{\mu + c^2} \mu_c V^0 \mu \right)$$

$$+ E^D \cdot c \left(V^0 \times H + \frac{1}{\mu + c^2} H \times V^0 \mu \right).$$  \hspace{1cm} (76)

The quantities $s^D, \rho^D, \rho_c^D, Q^D, g^{\text{tot},D}, \xi^D$ satisfy the same relations as before, see Eqs. (42)–(46). The dissipative fluxes are again obtained in an expansion of the entropy production $R$, Eq. (76).

The two Maxwell Eqs. (31) and (38) are not part of the hydrodynamic theory; rather, they simply define the quantities $\rho_c^D$ and $j_c^D$.

$$j_{c}^D + \rho_c^D v = c V \times H + \partial_c (v \times H)/c,$$

$$\rho + \rho_c^D := -V \cdot (v \times H)/c.$$  \hspace{1cm} (77)

Note that $H^0 \sim V \times E^0$ has been set to zero, while $\rho^D$ is given by Eq. (42).

4. Boundary conditions

A solution of the hydrodynamic equations is possible only if in addition to the equations we also have the appropriate initial and boundary conditions. The latter are obtained from the bulk equations themselves. Therefore, the number and type of the boundary conditions depend on the two systems comprising the interface. The boundary conditions are best derived in the rest frame of the
interface (or more generally in the rest frame of the portion of the interface under consideration).

4.1. The dielectric–dielectric interface, without phase transition

We shall first consider the simpler case in which no mass current may cross the interface, i.e. in the absence of the possibility of a phase transition, such as given at an interface made from two different substances.

The first boundary condition is the continuity of the normal component of the energy flux,

$$\Delta(Q_n + Q^n_D) = 0,$$

(78)

where the subscript $n$ denotes the component normal to the interface. This condition is obtained by integrating the energy conservation,

$$e^{\text{tot}} + V \cdot (Q + Q^D) = 0,$$

over an infinitesimally thick slab around the stationary interface. Physically, it simply implies that the same amount of energy enters and leaves the interface.

The notation, here and below, is given as

$$DA \equiv A(r - r_{sf} \rightarrow - 0)$$

$$- A(r - r_{sf} \rightarrow + 0) \equiv A_1 - A_2,$$

(79)

$$A \equiv \langle A \rangle \equiv \frac{1}{2}(A_1 + A_2),$$

(80)

where $r_{sf}$ is a given point on the surface. The normal vector $n$ hence points into region 2. Also,

$$\Delta(AB) \equiv A_1B_1 - A_2B_2 = A\Delta B + B\Delta A.$$ (81)

All vectors are divided into a tangential and a normal component, say

$$v_i = v - (v \cdot n)n,$$

$$v_n = (v \cdot n)n.$$ (82)

(83)

The lack of mass transfer implies

$$\rho_1v_{1,n} = j^D_{1,n}, \quad \rho_{c,1}v_{1,n} = j^D_{c,1,n}$$

$$\rho_2v_{2,n} = j^D_{2,n}, \quad \rho_{c,2}v_{2,n} = j^D_{c,2,n}.$$ (84)

Because $j^D_{1,n}$ and $j^D_{2,n}$ are relativistically small quantities, so are $v_{1,n}$ and $v_{2,n}$, hence also $j^D_{c,1,n}$ and $j^D_{c,2,n}$.

Three more boundary conditions follow from the conservation of momentum; one per component. In comparison to the conservation of energy, however, there is a complication arising from the possibility of an isotropic surface pressure [24].

$$\Pi^D_{ij} \equiv \Pi_{ij} - \alpha_{sf}(\delta_{ij} - n_in_j)d(r - r_{sf}),$$ (85)

where $\alpha_{sf}$ is the surface energy density of Ref. [24]. After some algebra, we obtain

$$\Delta(\Pi_{nn} - \Pi^n_D) = \alpha_{sf}\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$ (86)

$$\Delta(\Pi_{i,i} - \Pi^n_D)\mathbf{n}_{1,i} = - t_1 \cdot \nabla \alpha_{sf},$$ (87)

$$\Delta(\Pi_{i,i} - \Pi^n_D)\mathbf{n}_{2,i} = - t_2 \cdot \nabla \alpha_{sf},$$ (88)

where

$$\Pi_{i,i} = \Pi_{ij}\mathbf{n}_1(\delta_{ii} - n_in_i),$$

$$\Pi^n_{i,i} = \Pi^n_{ij}\mathbf{n}_1(\delta_{ii} - n_in_i),$$ (89)

$R_1, R_2$ denote the curvature radii, and $t_1, t_2$ the attendant principle directions. The radii are positive if they point into region 1, and the bulge is toward region 2.

From the Maxwell equations, we deduce the boundary conditions,

$$\Delta(B_n + B^n_D) = 0,$$ (90)

$$\Delta(E_i + E^n_D) = 0,$$ (91)

$$\Delta(D_n + D^n_D) = - \sigma_{sf},$$ (92)

$$\Delta(H_i + H^n_D) = \mathbf{n} \times \mathbf{j}_{el, sf}/c,$$ (93)

where $\sigma_{sf}$ denotes the surface charge density and $\mathbf{j}_{el, sf}$ the surface current, as yet undetermined.

Combining Eqs. (78) and (84), we obtain

$$0 = \Delta(Q_n + Q^n_D + (\mu + c^2)\rho v_n - (\mu + c^2)\rho v_n)$$ (94)

which in conjunction with Eqs. (49), (15) and (45) implies

$$0 = \Delta[(T_s + v \cdot g)v_n - \Pi^n_D(\mathbf{v}_n - T_f)v_n]$$

$$+ c(E \times H + E^D \times H^n_D + E^0 \times H^n_D) \cdot \mathbf{n}. $$ (95)
If the interface is curved, we shall from here on choose the inertial system \( \mathbf{v}_n = 0 \) (which gets rid of a term \( \sim \mathbf{v}_n \mathbf{z}_{st} \) in \( R^{st} \)), otherwise, \( \mathbf{v}_n \) is arbitrary. As \( \mathbf{v}_n \) is negligibly small in the absence of phase transition, Eq. (95) is quickly converted into the surface entropy production,

\[
R^{sf} \equiv - T \Delta f_n = f_n \Delta T - \Pi^{D}_{n} \Delta v_{n,i,j}
\]

\[
+ c \Delta (E \times H + E^{D} \times H^{0} + E^{0} \times H^{D}) \cdot n
\]

with \( f_n \equiv (\mathbf{v}_n - j^{D}_{n}) \), here and below. The surface entropy production \( R^{sf} \equiv - T \Delta f_n \) is the difference between the entropy current exiting and entering the surface. Just as its bulk counterpart, it is positive-definite, invariant under time inversion and vanishes in equilibrium. Some more algebra then yields

\[
R^{sf} = f_n \Delta T + (c(n \times E^{D}) \cdot \Delta H + c(\mathbf{H}^{D} \times n) \cdot \Delta E
\]

\[
+ ( - \Pi^{D}_{m} + \Pi^{D}_{j} B_{n} + E^{D}_{j} D_{n} + \frac{1}{2} \sigma_{st} \Delta E_{j}) \Delta v_{n,i,j}
\]

\[
+ [n \times j_{el,st}] \cdot (n \times E^{0}),
\]

(97)

a nice sum of independent pairs of force and fluxes. Taking them to be proportional to each other, they represent another seven boundary conditions, and determine the surface current \( j_{el,st} \).

One symmetry element peculiar to the interface is the invariance of \( R^{sf} \) under the simultaneous operation \( n \rightarrow - n \) and \( \Delta \rightarrow - \Delta \). Taking this and the isotropy into account, we find

\[
f_n = \kappa_{s} \Delta T,
\]

(98)

\[
\begin{gathered}
(n \times E^{D}) \\
(n \times j_{el,st})
\end{gathered}
= \begin{pmatrix}
\beta_{s} & \gamma_{s} & c \Delta H_{i} \\
\tilde{\gamma}_{s} & \sigma_{s} & n \times E^{0}
\end{pmatrix}.
\]

(99)

\[
- \Pi^{D}_{i,j} \equiv - \Pi^{D}_{i,j} + \Pi^{D}_{j} B_{n} + E^{D}_{j} D_{n} + \frac{1}{2} \sigma_{st} \Delta E_{i,j},
\]

(100)

where

\[
\begin{pmatrix}
- \Pi^{D}_{i,j} \\
(H^{D} \times n)
\end{pmatrix} = \begin{pmatrix}
\eta_{s} & \tilde{\eta}_{s} & \Delta v_{i,j} \\
\tilde{\eta}_{s} & \tilde{\eta}_{s} & \tilde{\eta}_{s}
\end{pmatrix},
\]

\[
\begin{pmatrix}
\beta_{s} & \gamma_{s} & c \Delta H_{i} \\
\tilde{\gamma}_{s} & \sigma_{s} & n \times E^{0}
\end{pmatrix}.
\]

These are altogether 21 boundary conditions. They suffice to determine (i) all (outgoing) hydrodynamic modes of the dielectric medium, nine for each side; (ii) the normal components of \( \mathbf{B} + \mathbf{B}^{D} \) and \( \mathbf{D} + \mathbf{D}^{D} \), (iii) the lab velocity. See Ref. [25] for a more detailed consideration (in single-component liquids).

4.2. The dielectric–dielectric interface, with phase transition

Mass and concentration currents across the interface renders the consideration slightly more complicated.

The 12 continuity conditions that remain are as follows:

\[
\Delta (Q_{n} + Q^{D}_{n}) = 0,
\]

(103)

\[
\Delta (\rho c v_{n} - j^{D}_{c,n}) = 0,
\]

(104)

\[
\Delta (\rho_{c} v_{n} - j^{D}_{c,n}) = 0
\]

(105)

\[
\Delta (\Pi_{nn} - \Pi^{D}_{nn}) = P_{sf},
\]

(106)

\[
\Delta (\Pi_{i,l} - \Pi^{D}_{i,l}) t_{1,i} = - \mathbf{t}_{1} \cdot \mathbf{v}_{z_{sf}},
\]

(107)

\[
\Delta (\Pi_{i,l} - \Pi^{D}_{i,l}) t_{2,i} = - \mathbf{t}_{2} \cdot \mathbf{v}_{z_{sf}}
\]

(108)

\[
\Delta (B^{D}_{n} + B_{n}) = 0.
\]

(109)

\[
\Delta (E^{D}_{c} + E^{D}) = 0,
\]

(110)

\[
\Delta (D^{D}_{n} + D_{n}) = - \sigma_{sf},
\]

(111)

\[
\Delta (H^{D}_{n} + H^{D}) = n \times j_{el,st}/c.
\]

(112)

The remaining nine boundary conditions must now be deduced from \( R^{sf} \). Combining Eqs. (103) and (104), we obtain

\[
0 = \Delta (Q_{n} + Q^{D}_{n}) - (\mu + c^{2}) \Delta (\rho v_{n} - j^{D}_{n})
\]

\[
= \Delta (Q_{n} + Q^{D}_{n} + (\mu + c^{2}) j^{D}_{n} - (\mu + c^{2}) \rho v_{n})
\]

\[
+ \langle \rho v_{n} - j^{D}_{n} \rangle \Delta \mu,
\]

(113)

which in conjunction with Eqs. (49), (15) and (45) leads to

\[
0 = \Delta [(T s + \mu_{c} \rho_{c} + \mathbf{v} \cdot \mathbf{g}) v_{n} - \Pi^{D}_{n} v_{j} - T j^{D}_{n}
\]

\[
- \mu j^{D}_{c,n} + c (\mathbf{E} \times \mathbf{H} + \mathbf{E}^{D} \times \mathbf{H}^{0}) + \mathbf{E}^{0}
\]

\[
\times \mathbf{H}^{D} \cdot \mathbf{n} \rangle \langle \rho v_{n} - j^{D}_{n} \rangle \Delta \mu.
\]

(114)
Employing Eq. (105), we have
\[
R^\text{ef} = -T\Delta f_n = f_n\Delta T + \langle \rho v_n - j_n^D \rangle \Delta \mu
+ \langle \rho_c v_n - j_n^D \rangle \Delta \mu_c - \Delta(v_j\Pi_{jn}^D) + \Delta(v\cdot g v_n)
+ c\Delta(E \times \mathbf{H} + E^D \times H^0 + E^0 \times H^0) \cdot \mathbf{n},
\]
(115)
which is more suitably written as
\[
R^\text{ef} = f_n\Delta T + \langle \rho_v - j_n^D \rangle \Delta \mu_c
+ \left[ \langle v_n g_j \rangle - \Pi_{jn}^D + \frac{1}{2}\sigma_{\text{sf}} \Delta E \right] \Delta \nu_{l,j}
+ \langle \rho v_n - j_n^D \rangle \Delta \mu^\text{eff} + c \left( \mathbf{n} \times \left( \mathbf{E}^D - \frac{v_n}{c} \times \mathbf{B} \right) \right)
\cdot \Delta \mathbf{H}^0
+ \left[ \mathbf{n} \times \mathbf{j}_{\text{sf}} \right] \cdot \left( \mathbf{n} \times \mathbf{E}^D \right),
\]
(116)
where
\[
\mu^\text{eff} = \left( \mu + \frac{v_n^2 g_n}{\rho v_n - j_n^D} - \frac{\Pi_{jn}^D v_n}{\rho v_n - j_n^D} \right).
\]
(117)
This \( R^\text{ef} \) yields nine boundary conditions and the value of the surface current \( j_{\text{sf}} \).

4.3. Interfaces involving conductors

The essential difference to the boundary conditions considered until now is the fact that the electric field \( D \) is no longer an independent variable. As a direct result, neither are the two boundary conditions,
\[
\Delta(D_n^D + D_n) = -\sigma_{\text{sf}},
\]
(118)
\[
\Delta(H_i + H_i^0) = \mathbf{n} \times j_{\text{sf}}/c.
\]
(119)
This also affects the surface entropy production. Starting from
\[
0 = \Delta(Q_n + Q_n^D) - (\mu + c^2)\Delta(\rho v_n - j_n^D)
= \Delta(Q_n + Q_n^D) + (\mu + c^2)\Pi_{jn}^D - (\mu + c^2)\rho v_n
+ \langle \rho v_n - j_n^D \rangle \Delta \mu,
\]
(120)
we have for the conductor–conductor interface
\[
0 = \Delta\left[ (Ts + \mu c \rho_c + v \cdot g) v_n - \Pi_{jn}^D v_j - T f_j^D \right.
- \mu c j_{c,n} + c(E \times \mathbf{H} + E^D \times H^0) \cdot \mathbf{n}
+ \langle \rho v_n - j_n^D \rangle \Delta \mu
\]
and hence
\[
R^\text{ef} = -T\Delta f_n = f_n\Delta T + \langle \rho v_n - j_n^D \rangle \Delta \mu_c
+ \langle \rho v_n - j_n^D \rangle \Delta \mu_c - \Delta(v_j\Pi_{jn}^D) + \Delta(v\cdot g v_n)
+ c\Delta(E \times \mathbf{H} + E^D \times H^0) \cdot \mathbf{n},
\]
(121)
or
\[
R^\text{ef} = f_n\Delta T + \langle \rho v_n - j_n^D \rangle \Delta \mu_c + \left[ \langle v_n g_j \rangle - \Pi_{jn}^D \right] \Delta \nu_{l,j}
+ \langle \rho v_n - j_n^D \rangle \Delta \mu^\text{eff} + c \left[ \mathbf{n} \times (\mathbf{E}^D + E^D) \right] \cdot \Delta \mathbf{H}^0.
\]
(123)
where \( \mu^\text{eff} \) is the same as before, see Eq. (117). This expression yields seven connecting conditions. So we have a total of 16 boundary conditions for the conductor–conductor interface. They suffice to determine all outgoing collective modes, seven for each side, the normal component of \( \mathbf{B} + \mathbf{B}^D \), and the lab velocity of the interface. (Note that the number of the collective modes is reduced in a conductor, because there are no sq-Modes [7]. Also, the electromagnetic waves are reduced to magnetic, diffusive modes.)

For the conductor–dielectric interface, Eq. (120) implies
\[
0 = \Delta\left[ (Ts + \mu c \rho_c + v \cdot g) v_n - \Pi_{jn}^D v_j - T f_j^D \right.
- \mu c j_{c,n} + c(E \times \mathbf{H} + E^D \times H^0) \cdot \mathbf{n}
+ \langle \rho v_n - j_n^D \rangle \Delta \mu - c(E_2^D \times H_2^D) \cdot \mathbf{n},
\]
(124)
where 2 denotes the dielectric, i.e. \( \mathbf{n} \) points into the dielectric. The entropy production is now
\[
R^\text{ef} = f_n\Delta T + \langle \rho v_n - j_n^D \rangle \Delta \mu_c
+ \left[ \langle v_n g_j \rangle - \Pi_{jn}^D \right] \Delta \nu_{l,j}
+ \langle \rho v_n - j_n^D \rangle \Delta \mu^\text{eff}
+ \left[ \mathbf{n} \times (\mathbf{E}^D + E^D) \right] \cdot c \Delta \mathbf{H}_c^D
+ (H_2^D + v_2/c \times D_2) \cdot c(E_2^D \times H_2^D).
\]
(125)
This $R^4$ yields nine boundary conditions. Ignoring cross terms, we have especially

$$n \times (E^D + E) = \beta_s c \Delta H^r, \quad \beta_s > 0. \quad (126)$$

Here we have altogether 18 boundary conditions. They suffice to determine the seven collective modes of the conductor and the nine collective modes of the dielectric, to fix the normal component of $B + B^D$ and to determine the lab velocity of the interface.

5. Two illustrative examples

The hydrodynamics is not simply a coarse-grained theory; it has a scale-invariance built in: Varying the resolution, the hydrodynamics remains valid. This feature makes the theory especially general and widely applicable. The same applies to the boundary conditions derived above, which are valid wherever the bulk theory is. Below are two examples for the vacuum–conductor interface, where we shall calculate some coefficients from two simple models. In both cases, we shall employ a higher resolution hydrodynamic theory to obtain the coefficient of the coarser-grained one.

More specifically, we shall in the first example calculate the coefficient in the boundary condition (126), and understand the discontinuity of the coarse-grained magnetic field $H$ as given by a highly conducting surface layer (and surface current) that is not explicitly accounted for in the given resolution. This is, of course, a trivial case, but it does serve to draw one scenario in which the discontinuity of the magnetic field is not surprising. The crucial point here is that the coarse-grained boundary conditions as presented above are applicable also for situations where a higher-resolution theory is not easily available, such as for a turbulent boundary layer. If we check our ambitions with respect to resolving the turbulence in detail, this layer and its turbulence enhanced magnetic diffusivity should be well accounted for by the same boundary condition.

The second example is less straightforward. Here, we employ the hydrodynamic theory for a dielectric as the higher-resolution description of the conductor. This sounds somewhat surprising at first and should be explained. The dielectric theory (including a finite conductivity) contains a stationary, collective mode (the sq-mode) that is given by an exponential decay of the electromagnetic field from the system’s surface [7]. Because the decay length shrinks with the conductivity, the hydrodynamic theory for conductors does not contain this mode, (though this may be amended by including higher order gradient terms). Nevertheless, the derived boundary conditions are such that the long ranged effects of the sq-mode are well accounted for, albeit on a grain size on which the sq-mode is no longer resolved. This statement is explicitly proven by comparing the dielectric with the conductor theory.

5.1. The highly conducting surface layer

We consider an infinitely long and conducting wire (region 1), of $r_0$, which is located in a vacuum (region 2), and subjected to a parallel, constant and homogeneous electrical field $E = E_0 e_z$.

5.1.1. Coarse-grained theory

Within the wire we have

$$E^D = \text{const } e_z = E_0 e_z, \quad (127)$$

and from Eq. (76)

$$\nabla \times H = \frac{1}{\beta c} E^D, \quad (128)$$

in vacuum we have

$$\nabla \times H = 0. \quad (129)$$

The connecting condition, Eq. (126), is

$$H_{1,\varphi} - H_{2,\varphi} = -\frac{E_0}{\beta c} e_\varphi, \quad (130)$$

($\varphi, r, z$: cylinder coordinates). So in the wire

$$H(r) = \frac{1}{2\beta c} E_0 r e_\varphi \quad (131)$$
and outside
\[ H(r) = \left( \frac{E_0 r_0}{2 \beta c} + \frac{E_0}{\beta c} \right) \frac{r_0}{r} e_\phi. \] (132)

5.1.2. High-resolution theory

We have, within the wire,
\[ \nabla \times E = 0, \quad c \nabla \times H = \sigma E \] (133)
and outside,
\[ \nabla \times E = 0, \quad \nabla \times H = 0 \] (134)
connected by
\[ \Delta H_i = 0, \quad \Delta E_i = 0. \] (135)

The solutions are, within the wire,
\[ E = E_0 e_z, \quad H(r) = \frac{\sigma}{2c} E_0 r e_\phi \] (136)
and outside
\[ H(r) = \frac{\sigma}{2c} E_0 r^2 e_\phi. \] (137)

Clearly, both theories agree if
\[ \beta = 1/\sigma, \quad 1/\beta_s = 0. \] (138)

To obtain a finite \( \beta_s \), we add a thin layer (say of thickness \( d = r_0/1000 \)) of highly conducting substance (say \( \sigma_L = 10^{18} \) s) to the wire (say \( \sigma = 10^{15} \) s).

The electric field remains constant throughout,
\[ E = E_0 e_z, \] (139)
so \( \Delta E_i = 0 \) and \( \nabla \times E = 0 \) are satisfied. As a result of employing \( c \nabla \times H = j_1 \) twice, we have within the wire,
\[ H(r) = \frac{\sigma}{2c} E_0 r e_\phi \] (140)
within the layer,
\[ H(r) = \left( \frac{\sigma_L}{2c} E_0 r + \frac{(\sigma - \sigma_L) E_0 r_0^2}{2c \frac{r_0}{r}} \right) e_\phi \] (141)
and in vacuum,
\[ H(r) = \left( \frac{\sigma_L}{2c} E_0 (r_0 + d) + \frac{(\sigma - \sigma_L) E_0 r_0^2}{2c} \right) \frac{1}{r} e_\phi \]
\[ = \left( \frac{\sigma}{2c} E_0 r_0 + \frac{\sigma_L d}{c} E_0 \right) \frac{r_0}{r} e_\phi + O \left( \frac{d}{r_0} \right). \] (142)

For the given numbers the second parenthesis has the same magnitude as the first.

5.1.3. Comparison

Choosing not to resolve the added layer, a comparison of Eq. (132) with Eq. (142) yields
\[ 1/\beta_s = \sigma_L d. \] (143)

5.2. Sq-Mode

5.2.1. The dielectric theory

The half space \( x < 0 \) is vacuum, and the half space \( x > 0 \) is a conductor, labeled 1 and 2, respectively. Region 1 has
\[ H \equiv 0, \quad E = E_0 e_y, \] (144)

The stationary Maxwell equations, in region 2,
\[ 0 = c \nabla \times (E + E^D), \] (145)
\[ 0 = j_1^D - c \nabla \times (H + H^D) \] (146)
with
\[ H^D = -\kappa c \nabla \times E, \quad E^D = \beta c \nabla \times H, \quad j_1^D = \sigma E \] (147)
are satisfied by the general solution
\[ E_x = E_c + A e^{-x/\lambda}, \quad H_x = -\frac{\sigma}{c} E_c x + H_c - \frac{\lambda}{\beta c} A e^{-x/\lambda}, \] (148)
where
\[ \lambda = \sqrt{\kappa \beta c^2 (1 + \beta \sigma)}. \] (149)
The constant amplitudes \( E_c, H_c \) and \( A \) are to be determined from the connecting conditions at
These are Eqs. (91), (93) and (97) (no surface current)
\[ H_i + H_i^0 = 0, \quad E_\parallel + E_\parallel^0 = E_0 e_y, \] (150)
\[ cH_i^0 = \zeta_i n \times E^0, \] (151)
where the last equation is a result of \( E^0 \equiv 0 \) and \( H^0 \equiv 0 \) in vacuum, leading to
\[ R^s = \ldots cH_i^0 \cdot (n \times E^0). \] (152)
So the special solution is
\[ E_y = \frac{1}{1 + \beta \sigma} E_0 + A e^{-\lambda x/\lambda_1}, \] (153)
\[ H_z = -\frac{\sigma}{c(1 + \beta \sigma)} E_0 x - \frac{\lambda}{\beta c} A e^{-\lambda x/\lambda} - \frac{\sigma \lambda}{c} A, \] (154)
\[ E_\parallel^0 = \frac{\beta \sigma}{1 + \beta \sigma} E_0 - A e^{-\lambda x/\lambda}, \] (155)
\[ H_\parallel^0 = \frac{2 \sigma \lambda}{\lambda_1} A e^{-\lambda x/\lambda}, \] (156)
with
\[ A = \frac{\sigma \beta \zeta_1}{(\alpha c^2/\lambda + \zeta_1)(1 + \beta \sigma)} E_0. \] (157)

5.2.2. The conducting theory

The stationary Maxwell equation (65) is reduced to
\[ \nabla \times E^0 = 0 \] (158)
with
\[ E^0 = \beta_c c \nabla \times H, \] (159)
c.f. Eq. (76). The connecting conditions are
\[ E_\parallel^0 = E_0 e_y, \] (160)
\[ n \times E^0 = -\beta_c c H_i, \] (161)
the first being a result of putting \( E_2 \equiv 0 \) and \( E_\parallel^0 \equiv 0 \) in Eq. (110), the second follows from Eq. (126). The solutions are
\[ E_\parallel^0 = E_0, \] (162)
\[ H_\parallel = -\frac{1}{\beta_c} E_0 x - \frac{1}{\beta_c} E_0. \] (163)
Now, comparing both results, we have finally
\[ \beta_c = \frac{1 + \beta \sigma}{\sigma}, \] (164)
\[ 1/\beta_c = \frac{\lambda \sigma^2 \beta \zeta_1}{(\alpha c^2/\lambda + \zeta_1)(1 + \beta \sigma)}. \] (165)

References