Advanced Time Series

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Overview of today’s exercise

• Programming rules in GAUSS
• Drawing random variables
• Simulation of an AR(1) process
ALWAYS start with small n!

ALWAYS output window.

ALWAYS start with simple calculations. Next enrich your program step by step!

ALWAYS from the inside and then go outside!

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Comment your program!!!

· Use useful and sensible names for your variables and programs!

· Create your own program collection!

Programming rules in GAUSS 2
Overview of today's exercise

· Estimating the parameters of a AR(1) process
· Estimating the parameters of a MA(1) process
· Using CML in GAUSS
· Writing the log-likelihood function AR(1) MA(1) - Theory
· Programming rules in GAUSS
Likelihood function AR(1) for 1st and 2nd observation

\[
\begin{align*}
Y_t &= c + \phi Y_{t-1} + \epsilon_t \\
\epsilon_t &\sim \text{ iidN}(0, \sigma^2)
\end{align*}
\]

density of first observation

\[
Y_1 \sim \text{N}\left(\frac{c}{1 - \phi}, \frac{\sigma^2}{1 - \phi^2}\right)
\]

\[
f_{Y_1}(y_1; c, \phi, \sigma^2) = \frac{1}{\sqrt{2\pi \sigma^2 / (1 - \phi^2)}} \exp\left[-\frac{(y_1 - \left[\frac{c}{1 - \phi}\right])^2}{2 \sigma^2 / (1 - \phi^2)}\right]
\]

Likelihood function AR(1) for 1st and 2nd observation
\[
(\theta; \mathbf{I}_{\mathbf{h}_1|\mathbf{z}_1}) \mathbf{I}_{\mathbf{z}_2|\mathbf{z}_1} f \cdot (\theta; \mathbf{I}_{\mathbf{h}_1|\mathbf{z}_1}) \mathbf{I}_{\mathbf{z}_1} f = (\theta; \mathbf{I}_{\mathbf{h}_1|\mathbf{z}_1}) \mathbf{I}_{\mathbf{z}_2|\mathbf{z}_1} f
\]

Joint density of first and second observation

\[
\left[ \frac{z o z}{z (z \mathbf{z}) -} \right] \exp \left[ \frac{z o y z}{I} \right] = (\theta; \mathbf{I}_{\mathbf{h}_1|\mathbf{z}_1}) \mathbf{I}_{\mathbf{z}_1} f
\]

\[
\mathbf{h}_1 \phi - c - \mathbf{z}_1 = z \mathbf{z}
\]

\[
\left[ \frac{z o z}{z (\mathbf{h}_1 \phi - c - \mathbf{z}_1) -} \right] \exp \left[ \frac{z o y z}{I} \right] = (\theta; \mathbf{I}_{\mathbf{h}_1|\mathbf{z}_1}) \mathbf{I}_{\mathbf{z}_1} f
\]

\[
(\mathbf{z} o (\mathbf{h}_1 \phi + c)) N \sim (\mathbf{h}_1 = \mathbf{I}_{\mathbf{z}_1|\mathbf{z}_1})
\]

\[
z \mathbf{z} + \mathbf{I}_{\mathbf{z}_1} \phi + c = z \mathbf{z}
\]

density of second observation
Writing the joint likelihood function AR(1)

\[
(\theta : \mathcal{I} \mid \mathcal{H}) \prod_{t=2}^{\tau} + (\theta : \{1\mid \mathcal{H}\}) \prod_{\mathcal{I}} = (\theta) \mathcal{F}
\]

Taking logs yields

\[
\left[ x_{2}^{\mathcal{O} \mathcal{Z}_{2}} \right] \text{d}x \exp \frac{x_{2}^{\mathcal{O} \mathcal{Z}_{2}}}{\mathcal{I}} = (\theta : \mathcal{I} \mid \mathcal{H}) \prod_{t=2}^{\tau} \left[ x_{2}^{\mathcal{O} \mathcal{Z}_{2}} \right] \text{d}x \exp \frac{x_{2}^{\mathcal{O} \mathcal{Z}_{2}}}{\mathcal{I}} = (\theta : \{1\mid \mathcal{H}\}) \prod_{\mathcal{I}}
\]

Writing the joint likelihood function AR(1)
Writing the log-likelihood function AR(1)

\[
\begin{align*}
\left( \begin{bmatrix}
\frac{\epsilon_2}{\sqrt{2\pi\sigma^2}} \\
\frac{\epsilon_2 - \epsilon_1}{\sqrt{2\pi\sigma^2}} \\
\epsilon_1
\end{bmatrix} \exp \frac{\epsilon_2^2}{2\sigma^2} \right) \log \exp \frac{\epsilon_2^2}{2\sigma^2} & \rightarrow (\theta; \epsilon_1 \mid \epsilon_2) \begin{pmatrix} \epsilon_2 - \epsilon_1 \end{pmatrix} \sigma^2 f_{\epsilon_2} \log \left( \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{\epsilon_2^2}{2\sigma^2} \right] \right) + \\
& \cdots \\
\left( \begin{bmatrix}
\frac{\epsilon_2}{\sqrt{2\pi\sigma^2}} \\
\frac{\epsilon_2 - \epsilon_1}{\sqrt{2\pi\sigma^2}} \\
\epsilon_1
\end{bmatrix} \exp \frac{\epsilon_2^2}{2\sigma^2} \right) \log \exp \frac{\epsilon_2^2}{2\sigma^2} & \rightarrow (\theta; \epsilon_1 \mid \epsilon_2) \begin{pmatrix} \epsilon_2 - \epsilon_1 \end{pmatrix} \sigma^2 f_{\epsilon_2} \log \left( \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{\epsilon_2^2}{2\sigma^2} \right] \right) + \\
& \log \text{deterministic} \rightarrow (\theta; \epsilon_1 \mid \epsilon_2) \begin{pmatrix} \epsilon_2 - \epsilon_1 \end{pmatrix} \sigma^2 f_{\epsilon_2} \log \left( \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{\epsilon_2^2}{2\sigma^2} \right] \right) = (\theta) \mathcal{F}
\end{align*}
\]
\( MA(1) \) Process

\[
\begin{align*}
\phi_0 \epsilon_t & = \epsilon_t \\
\epsilon_t & = \epsilon_{t-1} + \theta \epsilon_{t-1} + \eta_t
\end{align*}
\]
Writing the likelihood function \( MA(1) \)

\[
\begin{align*}
&\left[ \frac{z_0 z}{z(1-\theta \cdot \eta - \zeta \cdot \phi)} \right] \exp \left[ \frac{z_0 \mu z}{1} \right] = (\theta; 1-1-\theta \cdot \eta - \phi)_{1-1-\theta}^{1-\theta} \chi^2 \\
&1-\theta \cdot \eta - \phi = \zeta \\
&\left( \frac{z_0 z}{z(1-\theta \cdot \eta - \phi)} \right) \exp \left[ \frac{z_0 \mu z}{1} \right] = (\theta; 1-1-\theta \cdot \phi)_{1-1-\theta}^{1-\theta} \chi^2 \\
&(\zeta^{0 \cdot (1-\theta \cdot \eta + \phi)})_{\zeta} \sim 1-1-\theta \chi^2
\end{align*}
\]
Log likelihood function MA(1)

\[
L(\theta) = \log f(Y_T, Y_{T-1}, ..., Y_1 | \varepsilon_0 = 0) - \sum_{t=1}^{T} \varepsilon_t^2 \sigma^2
\]

\[
= (\theta : 0 = 0_3 | \lambda_4, \lambda_5, ..., \lambda_T) \log f(Y_T, Y_{T-1}, ..., Y_1 | \varepsilon_0 = 0, \lambda_4, \lambda_5, ..., \lambda_T)
\]

\[
= (\theta) K
\]
How to get $\varepsilon$ - Recursion

\[ \varepsilon_{t-1} \theta - \eta - \varepsilon_{t-1} \varepsilon = \varepsilon_{t-1} \]

\[ \vdots \]

\[ \varepsilon_{T} \theta - \eta - \theta_{T} = \varepsilon_{T} \]

\[ 0 = 0_{\varepsilon} \quad \text{with} \quad \eta - \varepsilon_{1} = 1_{\varepsilon} \]
Writing the conditional log-likelihood function MA(1)

\[
\left( \frac{\mathcal{Z}_T}{\mathcal{Z}_{(1)}^-} \right) \exp \left( \frac{\mathcal{Z}_{(1)}^\top}{I} \right) \log \left( (\theta : \cdot)_{0=0}^{Y_T, \ldots, Y_{T-1}} | \mathcal{Y} \right) \log \left( \mathcal{Y} \right) + 
\]

\[
\left( \frac{\mathcal{Z}_T}{\mathcal{Z}_{(1)}^-} \right) \exp \left( \frac{\mathcal{Z}_{(1)}^\top}{I} \right) \log \left( (\theta : \cdot)_{0=0}^{Y_T, \ldots, Y_{T-1}} | \mathcal{Y} \right) \log \left( \mathcal{Y} \right) + 
\]

\[
\left( \frac{\mathcal{Z}_T}{\mathcal{Z}_{(1)}^-} \right) \exp \left( \frac{\mathcal{Z}_{(1)}^\top}{I} \right) \log \left( (\theta : \cdot)_{0=0}^{Y_T, \ldots, Y_{T-1}} | \mathcal{Y} \right) \log \left( \mathcal{Y} \right) = (\theta) \mathcal{K}
\]
Numerical optimization of a function using an algorithm

• Input: function to be minimized and starting values for parameters, and data

• Output: vector of parameters and function value at minimum

Numerical optimization of a function using an algorithm

CML Procedure
CALL CML

start - a Kx1 vector of start values
e.g. \texttt{Gmmlike}

\texttt{fct} - the name of a procedure that returns the log-likelihood,
take vars = 0;

\texttt{vars} - character vector of labels selected for analysis

dataset - name of data matrix stored in memory

\texttt{INPUT}

\texttt{x,t,h, cov, recode} = \texttt{CML(dataset, vars, \texttt{fct}, \texttt{start})}
CML procedure-CALL

CALL CML(dataset,vars,start)

OUTPUT
rtcode - scalar, return code
cov - KxK matrix, covariance matrix of the parameters
g - Kx1 vector, gradient evaluated at x
f - scalar, function at minimum (mean log-likelihood)
x - Kx1 vector, estimated parameters

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Example:

CML procedure - GLOBALS
Some applications demand a small value in order to prevent convergence on a local minimum (local vs. global optima).

Important!!

When this criterion has been satisfied, CML will exit the iterations.

Default = 1e-5.

- \( \text{cml\_DirTol} \) is a scalar is a convergence tolerance for gradient of estimated coefficients.

CML global variables:

\( \text{cml\_DirTol} = 0.0000000001 \).
CML Global variables II

_cml_Algorithm = scalar indicator for optimization method

\(\text{cml\_Algorithm} = 1, \text{BFGS (Broyden, Fletcher, Goldfarb, Shanno)}\)
\(\text{cml\_Algorithm} = 2, \text{DFP (Davidon, Fletcher, Powell)}\)
\(\text{cml\_Algorithm} = 3, \text{NEWTON (Newton-Raphson)}\)
\(\text{cml\_Algorithm} = 4, \text{BHHH}\)

CML 8lobal\_cm\_Algorithm = scalar indicator for optimization method
CML Global variables III

- cml LineSearch:
  - 1 One
  - 2, STEPBT (default)
  - 3, HALF (step-halving)
  - 4, BRENT
  - 5, BHHSTEP
Two Sided Test: Example

P-value: 0.072

Observed value = -1.8
Reminder: P-value

\[ p\text{-val} = \sqrt{\text{diag(cov)}} \]

\[ t\text{-test} = \text{thx}./\text{se} \]

\[ p\text{-val} = 2*\text{cdftc(t-test, degrees of freedom)} \]

One-sided

Two-sided

General:

\[ p\text{-val} = 2*\text{cdftc(t-test, degrees of freedom)} \]

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A source file consists only of procedure code, no hard code should be written into a source file:

```cpp
#include mysourcefile.src;
```

The source file is then included into the program using:

```
#include mysourcefile.src;
```

To render your programs less confusing procedures can be written into source files.
Stationarity Test

 Null hypothesis: unit root (non stationarity), i.e., that $\rho = 1$

Dickey-Fuller Unit Root Test

$y_t - \rho y_{t-1} + \epsilon_t = \epsilon_t$
Simulation of Dickey-Fuller Test Statistic

Asymptotic distribution depends on specification of the true process (constant, time trend)

Inference requires simulation of asymptotic distribution

Non-standard asymptotic distribution of Unit Root processes

Simulation of Dickey-Fuller Test Statistic
Case 1

1. Simulate a Random Walk
   True Process: \( y_t = \phi y_{t-1} + \epsilon_t \)

2. Conduct an OLS regression
   Estimated Process: \( \hat{y}_t = \hat{\rho} y_{t-1} + \hat{\epsilon}_t \)

Calculate the t-statistic for the null hypothesis that the true value of \( \rho \) equals 1.
3. Simulate the test statistic

Run Step 2 $n=10000$ times and sort the t-values into quantiles.
Estimation of a GARCH(1,1) model to account for the stylized facts of financial return data.

Mean equation:

\[ y_t = c + \varepsilon_t \]

Variance equation:

\[ \sigma_t = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1} \]

(\( y_t \) is a log return time series)

A simple model to account for the stylized facts of financial return data.
Writing the conditional likelihood function GARCH(1,1)

\[ f(y_t | y_{t-1}, ..., y_0; \theta) = \frac{1}{\sqrt{2\pi h_t}} \exp\left[ -\frac{(y_t - c)^2}{2h_t} \right] \]

Conditional log likelihood function:

\[ L(\theta) = \sum_{t=1}^{T} \log f(y_t | y_{t-1}, ..., y_0; \theta) = -\frac{1}{2} \log(2\pi) - \frac{1}{2} \sum_{t=1}^{T} \log(h_t) - \frac{1}{2} \sum_{t=1}^{T} (y_t - c)^2 h_t \]

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