1. GMM inference

For the GMM estimator \( \hat{b}_{GMM} \) resulting from

\[
\arg\min_{\hat{b}} g_T(\hat{b})' W g_T(\hat{b})
\]

We have \( \hat{b}_{GMM} \xrightarrow{p} b \) and

\[
\sqrt{T}(\hat{b}_{GMM} - b) \xrightarrow{d} N(0, A\text{var}(\hat{b}_{GMM}))
\]

Where \( A\text{var}(\hat{b}_{GMM}) \) denotes the asymptotic variance covariance matrix. In a finite sample we use the approximation

\[
\hat{b}_{GMM} \sim N(b, \frac{A\text{var}(\hat{b}_{GMM})}{T})
\]

to test hypotheses about \( b \).

We have \( A\text{var}(\hat{b}_{GMM}) = (d'wd)^{-1}d'wSwd(d'wd)^{-1} \).

To compute \( A\text{var}(\hat{b}_{GMM}) \) you need to write

\[
d = \frac{\partial g_T(b)}{\partial b}'.
\]

\( g_T(b) \) is a vector valued function, i.e. it returns, for a given parameter vector \( b = (b_1, b_2, ..., b_k)' \), the vector of sample moments:

\[
\begin{pmatrix}
E_T(u_1^1(b)) \\
\vdots \\
E_T(u_N^N(b))
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{T} \sum_{t=1}^{T} u_1^1(b) \\
\vdots \\
\frac{1}{T} \sum_{t=1}^{T} u_N^N(b)
\end{pmatrix}
\]

\[
d = \frac{\partial g_T(b)}{\partial b}' \text{ is then}
\]

\[
\begin{pmatrix}
\frac{\partial E_T(u_1^1(b))}{\partial b_1} & \cdots & \frac{\partial E_T(u_1^1(b))}{\partial b_k} \\
\vdots & \ddots & \vdots \\
\frac{\partial E_T(u_N^N(b))}{\partial b_1} & \cdots & \frac{\partial E_T(u_N^N(b))}{\partial b_k}
\end{pmatrix}
N \times K
\]

Write \( d \) in detail for the GMM estimation framework of the consumption based model where

\[
m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma}.
\]
Use two moment restrictions for two asset returns $R_{t+1}^a$ and $R_{t+1}^b$:

$$E(m_{t+1}R_{t+1}^a - 1) = 0 \quad (7)$$

$$E(m_{t+1}R_{t+1}^b - 1) = 0 \quad (8)$$

What is $b$?
What is $u_t(b)$?
What is $E_t(u_t(b))$ and $g_T(b)$?
What is $\frac{\partial g_T(b)}{\partial b}$?

Write all in greatest detail!

You have succeeded in computing a consistent estimate of $Avar(\hat{b}_{GMM})$ for your GMM application.

$$Avar(\hat{b}_{GMM}) = \begin{pmatrix} 5 & 0.3 \\ 0.3 & 10 \end{pmatrix} \quad (9)$$

You have used $T = 100$ observations. Your GMM estimates are given by

$$\hat{\beta}_{GMM} = 0.8 \quad \hat{\gamma}_{GMM} = 0.1 \quad (10)$$

Compute an estimate of $Var(\hat{\beta}_{GMM})$ and $Var(\hat{\gamma}_{GMM})$.

Test the hypotheses

$$H_0 : \beta = 1 \quad H_0 : \gamma = 0$$
versus
$$H_A : \beta \neq 1 \quad H_A : \gamma \neq 0 \quad (11)$$

using the t-statistics

$$t_1 : \frac{\hat{\beta}_{GMM} - 1}{\sqrt{Var(\hat{\beta}_{GMM})}} \quad t_2 : \frac{\hat{\gamma}_{GMM}}{\sqrt{Var(\hat{\gamma}_{GMM})}} \quad (12)$$

$t_1$ and $t_2$ are approximately N(0,1) under the respective Null-Hypothesis.
2. Application of the $\delta$-method

Suppose you have obtained a GMM estimator for $b = \begin{bmatrix} \theta \\ \phi \end{bmatrix}$ i.e. $\hat{b} = \begin{bmatrix} \hat{\theta} \\ \hat{\phi} \end{bmatrix}$.

We have

$$\sqrt{T}(\hat{b} - b) \xrightarrow{d} N(0, \Sigma)$$

where $\Sigma$ is the asymptotic variance covariance matrix.

A consistent estimate of $\Sigma$, denoted $\hat{\Sigma}$, is given by

$$\hat{\Sigma} = \begin{pmatrix} 2 & 0.2 \\ 0.2 & 3 \end{pmatrix} \quad (14)$$

The sample has $T = 100$ observations.

Provide estimates of $\text{Var}(\hat{\theta})$ and $\text{Var}(\hat{\phi})$ using this information. The GMM estimates are $\hat{\theta} = 0.6$ and $\hat{\phi} = 0.4$

You are interested in testing whether

$$r = \frac{\hat{\phi}}{\hat{\phi} + \hat{\theta}} = 0.5 \quad (15)$$

Construct a suitable test statistic (again, a t-statistic). For this purpose compute an estimate of the variance of $\hat{r} = \frac{\hat{\phi}}{\hat{\phi} + \hat{\theta}} \ , \text{Var}(\hat{r})$, by using the $\delta$-method.

Hints:

$$a(b) = \frac{\phi}{\phi + \theta} = r \quad (16)$$

$$\hat{r} = a(\hat{b}) \xrightarrow{p} a(b) \quad (17)$$

$$\sqrt{T}(a(\hat{b}) - a(b)) \xrightarrow{d} N(0, A(b)\Sigma A(b)') \quad (18)$$

where $A(b) = \frac{\partial a(b)}{\partial b} = \begin{pmatrix} \frac{\partial a(b)}{\partial \phi}, \frac{\partial a(b)}{\partial \theta} \end{pmatrix}$

The test statistic is

$$t = \frac{\hat{r} - 0.5}{\sqrt{\text{Var}(\hat{r})}} \quad (19)$$

$t$ is approximately $N(0,1)$ under the Null Hypothesis that $r = 0.5$. 