Testing conditional predictions of asset pricing models: 
Scaled returns (managed portfolios) and scaled factors 
Readings: Cochrane (2002), Ch. 8, 10
We use instruments to test the conditional predictions of asset pricing models

\[ p_t = \mathbb{E} \left( m_{t+1}(b) \cdot x_{t+1} | I_t \right) \] or \[ 1 = \mathbb{E} \left( m_{t+1}(b) \cdot R_{t+1} | I_t \right) \]
or \[ 0 = \mathbb{E} \left( m_{t+1}(b) \cdot R^e_{t+1} | I_t \right) \]

I.i.e ”integrates out” conditional implications, lets us focus on unconditional implications of asset pricing model (model for S.D.F.):
\[ \mathbb{E} \left( m_{t+1}(b) \cdot R_{t+1} - 1 \right) = 0 \]

To test conditional implications write
\[ \mathbb{E} \left( Y_{t+1} | I_t \right) = 0 \] where \[ Y_{t+1} = (m_{t+1}(b) \cdot R_{t+1} - 1) \] or ...
\{Y_{t+1}\} a martingale difference sequence.

Properties of m.d.s include:
\[
\text{cov} \left( y_{t+1}, z_t \right) = 0 \quad \forall \quad z_t \in I_t
\]
\[ \mathbb{E} \left( y_{t+1} z_t \right) = 0 \] since \( 1 \in I_t \)
Testable restrictions therefore: \[ \mathbb{E} \left[ (m_{t+1}(b) \cdot R_{t+1} - 1) z_t \right] = 0 \quad \forall \quad z_t \in I_t \]
The use of instruments has an economic interpretation: Can the model price “managed portfolios“?

\[ \tilde{x}_{t+1} = x_{t+1}^i z_t \] conceived as (payoff of) \textbf{managed portfolios}, i.e. artificial assets.

Example: \( z_t = \frac{d_t}{p_t} \) invest if \( z_t \uparrow \)

\( \tilde{x}_{t+1} \) conceived as another payoff with price \( z_t p_t \)

If model correct, it prices any asset, also mgt. portfolios.

\[
\frac{z_t p_t}{p(\tilde{x}_{t+1})} = \mathbb{E}_t (m_{t+1}(b) \cdot x_{t+1} z_t) \quad \text{or} \quad z_t = \mathbb{E}_t (m_{t+1}(b) \cdot R_{t+1} z_t)
\]

i.e.

\[
\mathbb{E}(z_t) = \mathbb{E}(m_{t+1} R_{t+1} z_t) \quad \text{or} \quad \mathbb{E}[(m_{t+1} R_{t+1} - 1) z_t] = 0
\]
To test the conditional implications you simply “blow up“ the number
of assets by including meaningful managed portfolios and proceed as
before.

Practice: $N$ assets, $M$ instruments
$M$ moment restrictions

$$
\mathbb{E}\left(\left[m_{t+1} (b) R_{t+1} - 1\right] \otimes z_t\right) = 0
$$

With two assets and two instruments $z_t = (1, z_1^1)'$

$$
\mathbb{E} \left[ \begin{array}{c}
m_{t+1} (b) R_{t+1}^a - 1 \\
m_{t+1} (b) R_{t+1}^b - 1 \\
(m_{t+1} (b) R_{t+1}^a - 1)z_1^1 \\
(m_{t+1} (b) R_{t+1}^b - 1)z_1^1 \\
\end{array} \right] = 0
$$

or, emphasizing the managed portfolio interpretation

$$
\mathbb{E}(m_{t+1} (b) R_{t+1} \otimes z_t - 1 \otimes z_t) = 0
$$

$$
\mathbb{E}(m_{t+1} (b) x_{t+1} \otimes z_t - p_t \otimes z_t) = 0
$$
You should include economically meaningful instruments (managed portfolios)

- $p = \mathbb{E}(m.x)$ should price any asset, also managed portfolios

- if model prices all managed portfolios, conditional asset pricing model true.

- select few selected instruments (we also select few assets from millions available). New managed funds example

- Select meaningful instruments: Those affecting conditional distribution of returns

- Any $z_t \in I_t$ qualifies as an instruments, but if $\text{corr}((m_{t+1}R_{t+1}), z_t) = 0$ but $\text{corr}(R_{t+1}, z_t)$ small: weak instrument

- danger of using weak instruments (Hamilton, 1994, p. 426 for references)
Some more details and intuition on the choice of instruments

\[ p_t z_t = \mathbb{E}_t(m_{t+1} x_{t+1} z_t) \quad \text{resp.} \quad z_t = \mathbb{E}_t(m_{t+1} R_{t+1} z_t) \]

holds true trivially if \( \text{corr}((m_{t+1} R_{t+1} - 1), z_t) = 0 \)
but an interesting instrument implies \( \text{corr}(R_{t+1}, z_t) \neq 0 \) and/or
\( \text{corr}(m_{t+1}, z_t) \neq 0 \)

if \( \mathbb{E}_t(R_{t+1}) \uparrow \) when \( z_t \uparrow \)

then in

\[ 1 z_t = z_t \underbrace{\mathbb{E}_t(R_{t+1})}_{\uparrow} \underbrace{\mathbb{E}_t(m_{t+1})}_{\downarrow} + z_t \underbrace{\text{cov}_t(m_{t+1} R_{t+1})}_{\downarrow} \]
Is a conditional asset pricing model testable at all?

Most asset pricing models imply **conditional** moment restrictions

\[ 1 = \mathbb{E}(m_{t+1}(b_t) \cdot R_{t+1} | I_t) \]

e.g. CAPM \( m_{t+1} = a_t - b_t R_t^W \).

Parameters of factor pricing model vary over time.  
⇒ unconditioning via l.i.e. no longer possible:

\[ 1 = \mathbb{E}(m_{t+1}(b_t) \cdot R_{t+1} | I_t) \]

does NOT imply

\[ 1 = \mathbb{E}(m_{t+1}(b) \cdot R_{t+1}) \]

this is not repaired by using scaled returns. GMM estimation no possible.

Hansen and Richard critique: CAPM (or other factor model) is not testable.
Scaled factors are a partial solution to the problem

With linear factor model

\[ m_{t+1} = b_t' f_{t+1} \]

use of "scaled factors" a partial solution:

"Blow up" number of factors by scaling factors with \((M \times 1)\) instruments vector \(z_t\) observable at \(t\)

\[ m_{t+1} = b' (f_{t+1} \otimes z_t) \]

Unconditioning via l.i.e. and GMM procedure as above.
Time varying parameters lead to scaled factors (single factor case)

**Motivation**

Consider linear one factor model \( m_{t+1} = a_t + b_t f_{t+1} \) \((f_{t+1} \text{ scalar})\)

Assume Parameters vary with \( M \times 1 \) instruments vector \( z_t \).

\[
  m_{t+1} = a(z_t) + b(z_t) f_{t+1}
\]

With linear functions

\[
  a(z_t) = a' z_t \quad \text{and} \quad b(z_t) = b' z_t
\]

\[
  \Rightarrow m_{t+1} = a' z_t + (b' z_t) f_{t+1}
\]

Mathematically equivalent to

\[
  m_{t+1} = \tilde{b}' (\tilde{f}_{t+1} \otimes z_t)
\]

where \( \tilde{b} = \begin{bmatrix} a \\ b \end{bmatrix} \), \( \tilde{f}_{t+1} = \begin{bmatrix} 1 \\ f_{t+1} \end{bmatrix} \)

Number of parameters to estimate \( 2 \cdot M \)
Time varying parameters lead to scaled factors (multi factor case)

Multi-factor case:

\[ m_{t+1} = b_t' f_{t+1} \]

Again: Time varying parameters linear functions of \( M \times 1 \) vector of observables \( z_t \).

\[ m_{t+1} = b(z_t)' f_{t+1} \quad \text{with} \quad b(z_t) = B z_t \]

Equivalent to \( m_{t+1} = \tilde{b}'(f_{t+1} \otimes z_t) \) where \( \tilde{b} = vec(B) \)

In practical application some elements of \( B \) may be set to zero.
Using scaled factors we can condition down and apply GMM

Conditioning down and GMM estimation possible

\[
\mathbb{E}_t \left( \left( \tilde{b}'(f_{t+1} \otimes z_t) \right) R_{t+1} \right) = 1 \quad \text{l.i.e. } \Rightarrow \mathbb{E}\left( \left( \tilde{b}'(f_{t+1} \otimes z_t) \right) R_{t+1} - 1 \right) = 0
\]

Scaled factors and managed portfolios can be combined. (\(z_t\) might be the same).

\[
\Rightarrow \mathbb{E}(\tilde{b}'(f_{t+1} \otimes z_t) R_{t+1} - 1] \otimes z_t) = 0
\]

- Inclusion of conditioning information as managed portfolios (scaled returns, increases number of test assets.
- Scaled factors increase number of unknown parameters
Cochranes (1996) CAPM with scaled factors

\[ f = \begin{pmatrix} 1 \\ RW \end{pmatrix} \]
\[ z_t = \begin{pmatrix} 1 \\ P \\ D \\ \text{term} \end{pmatrix} \]
\[ B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \]

\[ f \otimes z = \begin{pmatrix} 1 \\ RW \\ P \\ D \\ R^W \cdot P \\ D \\ \text{term} \\ R^W \cdot \text{term} \end{pmatrix} \]
\[ \tilde{b} = \left( b_{11}, b_{21}, b_{12}, b_{22}, b_{13}, b_{23} \right)' \]

\[ m = \tilde{b}'(f \otimes z) = b_{11} + b_{12} \frac{P}{D} + b_{13} \text{term} + b_{21} R^W + b_{22} R^W \cdot \frac{P}{D} + b_{23} R^W \cdot \text{term} \]

In application Cochrane (1996) restricts \( b_{12} \) and \( b_{13} \) to zero