We often use a convenient power utility function (1)

\[ u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma} \]

\[ \lim_{\gamma \to 1} \left( \frac{1}{1-\gamma} c_t^{1-\gamma} \right) = \ln(c_t) \]

\[ u'(c_t) = c_t^{-\gamma} \]

\[ \frac{dc_t}{dc_{t+1}} = \frac{\beta u'(c_{t+1})}{u'(c_t)} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \]

Negative of marginal rate of substitution

utility \( u(c_t) \)

\[ 0.3 \quad 0.5 \quad 0.8 \]

parameter \( \gamma \): increasing concavity of utility function
We often use a convenient power utility function (2)

abs(marginal rate of substitution)

subj. discount factor $\beta$: 0.9

parameter $\gamma$:

$0.3 \quad 0.5 \quad 0.8$

ratio $(c_{t+1} / c_t)$
Measuring absolute risk aversion (1)

Find the risk premium \( \pi_Z \), such that investor is indifferent to receiving the risk \( Z \) and receiving the non-random amount \( \mathbb{E}(Z) - \pi_Z \)

\[
\begin{align*}
\text{wealth (non-random)} & \quad \downarrow \quad \downarrow \quad \downarrow \\
\text{expected value of} & \quad \text{expected utility} \\
\text{risky payment} & \quad \pi_Z \\
u \{w + \mathbb{E}(Z) - \pi_Z (w)\} & = \mathbb{E}[u(w + Z)]
\end{align*}
\]

Assumptions:
\[
\mathbb{E}(Z) = \mu_Z = 0, \quad \sigma^2_Z \rightarrow 0, \quad \mathbb{E}\left[\left|(Z - \mu_Z)^3\right|\right] = o\left(\sigma^2_Z\right)
\]

neutral risk

Taylor expansion of the left hand side:
\[
u(w - \pi_Z) = u(w) - \pi_Z \cdot u'(w) + \frac{1}{2!} \pi_Z^2 \cdot u''(w) + \ldots
\]
\[
= u(w) - \pi_Z \cdot u'(w) + O\left(\pi^2_Z\right)
\]
Measuring absolute risk aversion (2)

Taylor expansion of the right hand side:

\[
\mathbb{E}[u(w + Z)] = \mathbb{E}\left[u(w) + Z \cdot u'(w) + \frac{1}{2!}Z^2u''(w) + O\left(Z^3\right)\right]
\]

\[
= u(w) + \mathbb{E}(Z) \cdot u'(w) + \frac{1}{2}\sigma_Z^2u''(w) + o\left(\sigma_Z^2\right)
\]

Equating both sides:

\[
u(w) - \pi_Z \cdot u'(w) = u(w) + \frac{1}{2}\sigma_Z^2u''(w) + o\left(\sigma_Z^2\right)
\]

Risk premium \[\pi_Z = \frac{1}{2}\sigma_Z^2 \cdot r(w) + o\left(\sigma_Z^2\right)\]

Arrow-Pratt measure of absolute risk aversion \[r(w) \triangleq -\frac{u''(w)}{u'(w)} = -\frac{d}{dw}\log\left(u'(w)\right)\]

the more concave the utility function, the higher the risk aversion
Measuring relative (or proportional) risk aversion (1)

Find the proportional risk premium $\pi^*_Z$, such that investor is indifferent to receiving the proportional risk $w \cdot Z$ and receiving the non-random amount $\mathbb{E}(w \cdot Z) - w \cdot \pi^*_Z$

$$u \left\{ w + \mathbb{E}(w \cdot Z) - w \cdot \pi^*_Z (w) \right\} = \mathbb{E} [u (w + w \cdot Z)]$$

as above:

$$\mathbb{E}(Z) = \mu_Z = 0, \quad \sigma^2_Z \to 0, \quad \mathbb{E} \left[ \left| (Z - \mu_Z)^3 \right| \right] = o \left( \sigma^2_Z \right)$$

Taylor expansion of the left hand side:

$$u \left( w - w \cdot \pi^*_Z \right) = u (w) - w \pi^*_Z \cdot u' (w) + \frac{1}{2!} w^2 \pi^*_Z \cdot u'' (w) + ...$$

$$= u (w) - w \pi^*_Z \cdot u' (w) + O \left( w^2 \pi^*_Z \right)$$
Measuring relative (proportional) risk aversion (2)

Taylor expansion of the right hand side:

$$\mathbb{E} [u(w + wZ)] = \mathbb{E} \left[ u(w) + wZ \cdot u'(w) + \frac{1}{2} w^2 Z^2 u''(w) + O \left( Z^3 \right) \right]$$

$$= u(w) + w \mathbb{E}(Z) \cdot u'(w) + \frac{1}{2} w^2 \sigma_Z^2 u''(w) + o \left( \sigma_Z^2 \right)$$

$$= u(w) + \frac{1}{2} w^2 \sigma_Z^2 u''(w) + o \left( \sigma_Z^2 \right)$$

Equating both sides:

$$u(w) - w \pi^*_Z \cdot u'(w) = u(w) + \frac{1}{2} w^2 \sigma_Z^2 u''(w)$$

Risk premium $$\pi^*_Z(w) = \frac{1}{2} \sigma_Z^2 \cdot r^*(w) + o \left( \sigma_Z^2 \right)$$

Arrow-Pratt measure of proportional risk aversion

$$r^*(w) = -w \frac{u''(w)}{u'(w)} = w \cdot r(w)$$
The specification of the utility function implies a close link between risk aversion and intertemporal elasticity of substitution.

\[
- \frac{u''(c_t)}{u'(c_t)} = \frac{\gamma}{c_t} \quad \text{absolute risk aversion coefficient}
\]

\[
- \frac{c_t \cdot u''(c_t)}{u'(c_t)} = \gamma \quad \text{relative risk aversion coefficient}
\]

\[
\sigma \equiv -\frac{\frac{d}{c_t+1}}{\frac{dMRS}{MRS}}
\]

\[
\frac{1}{\sigma} = -\frac{dMRS}{d\left(\frac{c_{t+1}}{c_t}\right)} \cdot \frac{\left(\frac{c_{t+1}}{c_t}\right)}{MRS} = -\gamma \cdot \beta \cdot \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma-1} \cdot \left(\frac{c_{t+1}}{c_t}\right) \\
\sigma = \frac{1}{\gamma}
\]