1 Linear Algebra

PD Dr. Thomas Dimpfl

Chair of Statistics, Econometrics and Empirical Economics
1.9 Quadratic forms and sign definitness
Readings

- Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne Strøm. *Further Mathematics for Economic Analysis*. Prentice Hall, 2008 Chapter 1
Online Resources

MIT course on Linear Algebra (by Gilbert Strang)

- Lecture 26: Symmetric matrices and positive definiteness
  https://www.youtube.com/watch?v=umt6BB1nJ4w
- Lecture 27: Positive definite matrices and minima – Quadratic forms
  https://www.youtube.com/watch?v=vF7eyJ2g3kU
1.9 Quadratic forms and sign definiteness

Definitions

- Degree of a polynomial
- Form of $n$th degree
- special case: quadratic form

$$Q(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$
A quadratic form $Q(x_1, x_2)$ for two variables $x_1$ and $x_2$ is defined as

$$Q(x_1, x_2) = x' A x = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} x_i x_j$$

where $a_{ij} = a_{ji}$ and, thus,

with the symmetric coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$
1.9 Quadratic forms and sign definitness

Graph of the positive definite form $Q(x_1, x_2) = x_1^2 + x_2^2$

Graph of the positive semidefinite form $Q(x_1, x_2) = (x_1 + x_2)^2$

Graph of the negative definite form $Q(x_1, x_2) = -x_1^2 - x_2^2$

Graph of the negative semidefinite form $Q(x_1, x_2) = -(x_1 + x_2)^2$

Graph of the indefinite form $Q(x_1, x_2) = x_1^2 - x_2^2$
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The quadratic form associated with the matrix $A$ (and thus the matrix $A$ itself) is said to be

- **positive definite**, if $Q = x'Ax > 0$ for all $x \neq 0$
- **positive semi-definite**, if $Q = x'Ax \geq 0$ for all $x$
- **negative definite**, if $Q = x'Ax < 0$ for all $x \neq 0$
- **negative semi-definite**, if $Q = x'Ax \leq 0$ for all $x$

Otherwise the quadratic form is **indefinite**.

**Note**: For any quadratic matrix $A$ it holds that $x'Ax = x'Bx$ with $B = 0, 5 \cdot (A + A')$ symmetric.
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The quadratic form $Q(x)$ is

- positive (negative) definite, if all eigenvalues of the matrix $A$ are positive (negative): $\lambda_j > 0$ ($\lambda_j < 0$) $\forall j = 1, 2, \ldots, n$;

- positive (negative) semi-definite, if all eigenvalues of the matrix $A$ are non-negative (non-positive): $\lambda_j \geq 0$ ($\lambda_j \leq 0$) $\forall j = 1, 2, \ldots, n$ and at least one eigenvalue is equal to zero;

- indefinite, if two eigenvalues have different signs.
1.9 Quadratic forms and sign definitness

Properties of positive definite and positive semi-definite matrices

1) Diagonal elements of a positive definite matrix are strictly positive. Diagonal elements of a positive semi-definite matrix are nonnegative.

2) If $\mathbf{A}$ is positive definite, then $\mathbf{A}^{-1}$ exists and is positive definite.

3) If $\mathbf{X}$ is $n \times k$, then $\mathbf{X}'\mathbf{X}$ and $\mathbf{XX}'$ are positive semi-definite.

4) If $\mathbf{X}$ is $n \times k$ and $\text{rk}(\mathbf{X}) = k$, then $\mathbf{X}'\mathbf{X}$ is positive definite (and therefore non-singular).