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INTRINSIC GEOMETRY OF PERIPERSONAL SPACE

Master Thesis

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ABSTRACT

Physical space as a ubiquitous aspect of everyday experience is often thought to be in three linear dimensions, but modern physicists usually describe it alongside the time as an unlimited 4-dimensional link called spacetime. It has been historically discussed and experimentally shown that the physical space is different from the one that we visually perceive. This visually perceived space surrounding us is three-dimensional, but its intrinsic geometry is not Euclidean. This can be demonstrated by studying lines that are perceived as being straight, i.e., as being the shortest connection between two points. Such perceptual geodesics deviate significantly from the true straight lines of Euclidean space.

Luneburg in 1947 was the first to experiment the perceptual space geometry and concluded that space of binocular perception could be formulated in terms of negative Riemannian geometry of constant curvature as a description for visual space. Koenderink et.al., (2000) introduced a novel method to probe human optical space, to be able to address the Luneburg hypothesis that human optical space should be of constant (negative) curvature. They introduced a new method called exocentric pointing task and argued that the precious studies were concerned only with the near-binocular space while the properties of a richly structured visual space beyond near distance have received less attention. They demonstrated that this curvature changes from elliptic in near space to hyperbolic in far space while the plane becomes parabolic at very large distances.

In this thesis work we measure tangents to geodesic in the horizontal plane at eye-level by repeating the experiment paradigm discussed in aforementioned experiments and then extend the idea by measuring the geodesics in elevated level.

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Chapter I: Introduction

The concept of space, as an unbounded extent in which any object and event have relative direction and position that can be located with three numerical coordinates, has primary importance in recognition and perception of the physical environment. Physical space as a ubiquitous aspect of everyday experience is often thought to be in three linear dimensions, but modern physicists usually describe it alongside the time as an unlimited 4-dimensional link called spacetime. Although this term can refer to wide range of phenomena from astronomy and cosmology, mathematics, and even communications, in this thesis however we focus on the geometrical relationships between physical and visually perceived space to experiment this very old philosophical question which has not been satisfactorily answered yet. We start with briefly discussing the fundamental questions and concepts of visual space perception and highlighting the important aspects of distinction between physical and perceived spaces. Since this concept has been discussed for nearly 2000 years and starting from the 19^{th} century a plethora of research in this field has been introduced, in this chapter we attempt only to explain the most related concepts as well as to review the relevant literature to the focus of this thesis.

Space

The philosophical discussions over existence of the space returns to ancient times, in the inscriptions from Socrates in which he discussed the Greeks termed the word “khôra” (meaning space), or explanation of the word “topos” (meaning space) in Aristotle ‘s container view of space and place as an extendable limit. Later in the seventeenth century, Isaac Newton presented the definition of absolute space, which was always permanent and independent of the matter in it while Gottfried Wilhelm Leibniz ‘s idea about space was a collection of correlations between objects that were defined by their distance and direction. This debate between the Leibniz and the Newton concerning the status of space and time forms part of the essential background to one of the most critical views over space from a famous German philosopher and mathematician Immanuel Kant.

By the turn of eighteen century, Immanuel Kant presented, in his masterpiece “*Critique of Pure Reason,* that concept of space and time are not experimental concepts that are coming from experiences from outside world but are part of a systematic framework inside of human beings and are used to organize and structure all experiences in outside world. *He illustrated the limitations of* absolutist and rationalist conceptions of space and time and argued that space awareness is *synthetic*. According to Kant ‘s view, he did not believe the idea that space must be either a relationship or matter. Instead, he concluded that human does not discover space and time to be a world objective features but forced by us as part of a framework for structuring experience. However, there is still disagreement among philosophers as to whether space itself is an existence or whether it is a relation between entities or part of a conceptual framework.

Visual Space

The Space that we perceive through vision is our visual space. Euclid's geometrization of optics suggested that visual space was confined to a cone with the base at the limits of vision and the cone itself having its vertex at the eye (Burton 1945) . Reid proposed in the 18th century that there is a correspondence between visual space and the shape of the eyes and concluded that the visual space is spherical (Suppes 1977). Reid formulated his geometry of vision which is a geometry of monocular vision as: "Supposing the eye placed in the center of a sphere, every great circle of the sphere will have the same appearance to the eye as if it was a straight line; for the curvature of the circle being turned directly toward the eye, is not perceived by it. And, for the same reason, any line which is drawn in the plane of a great circle of the sphere, whether it be straight or curve, will appear to the eye." (for a detailed historical background see: Suppes, 1977).

Luneburg was the first to experiment the perceptual space geometry by integrating psychophysical evidence empirically from measurements made of human observers with mathematical modeling (Luneburg 1947). He proceeded on these assumptions: "We recognize, by binocular vision, that we are surrounded by a three-dimensional manifold of objects. These objects have, beside characteristic qualities of color and brightness, form, and localization. In a visual sensation we thus are not only immediately aware of a distribution of colors and brightness, but also of the fact that certain of these qualities are combined to unities, namely, objects, which have a definite geometrical form and a definite localization in a three-dimensional space. We shall call this space the visual space." (Luneburg 1947). Proceeding on these assumptions he mapped the resulting experimental data to a Riemannian manifold and was the first person who concluded that space of binocular perception could be formulated in terms of negative Riemannian geometry of constant curvature as a description for visual space (W. H. Rosar 2016). This model was later developed by (Blank 1958, 1978) who described only qualitatively the mapping of visually perceived space and physical space with the assumption of a constant curvature. Several other studies (i.e., Battro et.al., 1976; Shipley, 1957) also concluded that visual space is curved although a few authors doubted and criticized its hyperbolic nature. Foley (1972) for instance, demonstrated that perceived visual angles and perceived distances are the product of independent distinctive processes. His conclusion implies that perceived angles and distances do not constitute a single visual space. In addition, Indow (1991) later showed in his critical reviews of Luneburg's model that the results were very task- and subject-dependent and therefore could not quantitatively describe this departure from veridical.

Koenderink et.al., (2000) introduced a novel method to probe human optical space, to be able to address the Luneburg hypothesis that human optical space should be of constant (negative) curvature. They introduced a new method called exocentric pointing task (this will be discussed in detail in the following chapters) in a wide range of distances (2 –20 meters) and argued that the precious studies were concerned only with the near-binocular space while the properties of a richly structured visual space beyond near distance have received less attention (Beeson 2015). They demonstrated that this curvature changes from elliptic in near space to hyperbolic in far space while the plane becomes parabolic at very large distances. The results of this paper were confirmed in its succeeding research in 2003. The main motivation behind this thesis was first to measure tangents to geodesic in the horizontal plane at eye-level by repeating the experiment paradigm discussed in these two scientific papers and then to extend the idea by measuring the geodesics in elevated level.

Geometry of Space

Intrinsic vs. Extrinsic Geometry

Following the Kant's view, we discuss the intrinsic geometry of perceptual space which provides a global set of constraints, irrespective of the relation to the external environment, by which the judgments of a given observer are formally related to one another. In this thesis we do not investigate the extrinsic geometry which refers to the relationship between the structure of the observer's perception and the actual structure of physical space. Since in this study, we aim first to investigate the intrinsic geometry of peripersonal space and second to observe that to what extent it deviates from Euclidean geometry, in following a short review of important features of Euclidean as well as non-Euclidean geometries will be provided.

Euclidean Geometry

Euclidean geometry is the study of geometry based on definitions, terms, and the assumptions of the mathematicians Euclid (330 B.C.). It is been described as the geometry of the flat surfaces and it is credited with developing the first comprehensive deductive system of geometry. This plane geometry which is the study of lines and shapes on a flat surface can be illustrated only on a flat paper and is not capable of describing physical space such as curved space. Euclid proposed five postulates (Axioms) that create a basis for his geometry:

1. It is possible to draw a straight line from any point to any point.
2. If you have a straight line it is possible to extend in any direction to infinity.
3. It is possible to draw a circle given any center and a radius.
4. All right angles are equal (congruent).
5. Parallel postulate. If you have two straight lines, and a third line crossing them, and the sum of the interior angle measure of the two lines angles is less than 90 degrees, then if you extend the lines far enough, they will eventually cross on that side (e.g., Beeson 2015; Patel 2018).

Euclidean geometry is divided in two types which includes plane geometry, which is two-dimensional Euclidean geometry, and solid geometry, which is three-dimensional Euclidean geometry. This is a classical geometry which is used for modeling of the space of the physical world (Patel 2018). The application of Euclidean geometry is in many scientific areas namely, astronomy, mechanics, crystallography as well as some technical fields, such as geodesy, engineering, navigation, aerodynamics, and architecture. Despite its ubiquitous applications, the Euclidean geometry cannot work on non-flat surfaces. Unlike what initial four postulates of Euclid, the fifth postulate, the famous parallel postulate, revealed a lack intuitive appeal, and several mathematicians throughout history struggled to prove it (Trudeau 2001).

Non-Euclidean Geometry

Many attempts have been made to prove the fifth postulate using the other four postulates. All these attempts have failed. In the 19th century (Carl Friedrich Gauss, 1813; Ferdinand Karl Schweikart, 1818) it was shown that the fifth postulate is independent of the other postulates. It is possible to build a theory of geometry where the fifth postulate is not true. Such geometries are called non-Euclidean. In contrast to the Euclidean geometry, this geometry refers to certain types of geometry which differ from plane and solid geometry that had dominated the realm of mathematics for several centuries. Following the line of research in this study, we briefly overview the two of the main non-Euclidean geometries (elliptic and hyperbolic) which do not assume all of Euclid's postulates and formulate more precisely the more complex components of curves in space. The most relevant aspect of these different geometries discussed here is the surface on which they have been formulated. Obviously, the detailed review of these geometries is out of the scope of this thesis.

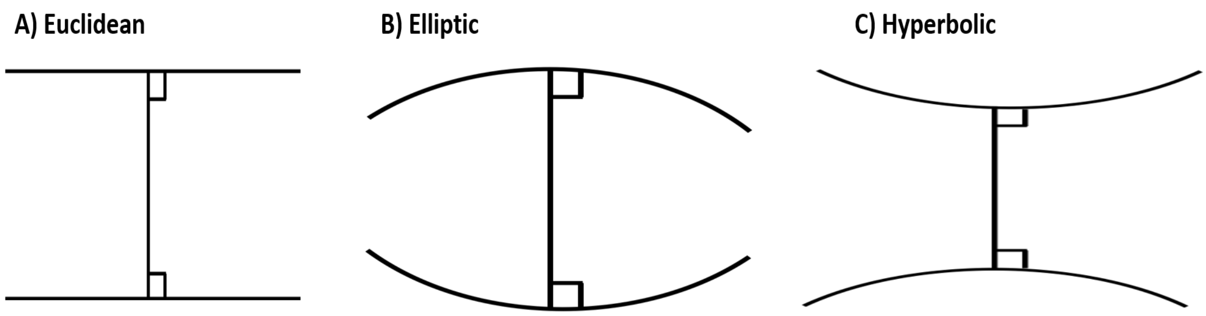


Figure 1 Behavior of lines with a common perpendicular in each of the three types of geometry (Figure adapted from https://en.wikipedia.org/wiki/Non-Euclidean_geometry)

Elliptic Geometry

An example of non-Euclidean geometry in which Euclid's parallel postulate does not hold is elliptic geometry. It is the simplest form of the infinite family of non-Euclidean geometries that were formulated by a German mathematician Bernhard Riemann, in his famous lecture in 1854 (Patel 2018). This geometry is formulated on the surfaces of (Riemann) Spheres meaning that straight line on the surface of a sphere would become an arc, and the longest line would equate to the diameter of the sphere. Navigation and astronomy are two main disciplines in which elliptic geometry is applied.

Hyperbolic Geometry

Another example of non-Euclidean geometry is the hyperbolic geometry which is the geometry of saddle or pseudospherical surfaces or simply surfaces with a constant negative curvature. This geometry which is also called Bolyai-Lobachevskian satisfies all of Euclid's postulates except the parallel postulate. In hyperbolic geometry, the sum of angles of a triangle is less than 180 degrees, and triangles with the same angles have the same areas.

Visual perception

Visual perception is defined as the mental organization and interpretation capability of the visual sensory information with the intent of attaining awareness and understanding of the objects and events surrounding us (Gibson, 1950). As we move and explore the environment, the visual stimulation in our eye is constantly changing. In general, any action that we make could potentially change our visual sensory information and consequently our visual perception. To understand the mapping between visual space and the physical space, the connection between the action that we make and the space that we visually perceive as well as the perceptual variables or cures and their couplings are important. Visual space is classically defined as an abstract structure independent of its contents. Aligned with the focus of this thesis however, we only briefly review the main aspects and features of visual stimuli that are crucial in determining the structure of visual space.

Space perception

Space perception is the knowledge of the distance, direction, location, scale, and shape of an object in the environment. All senses are fused together to form a unit perception of space that allows persons to orient themselves to their environment and create an awareness of reality. The brain acquires and processes this sensory information about the spatial layout of the environment to make us capable of navigating in the environment. In three-dimensional perception of an object, space perception is important for orientation and movement by providing depth and distance.

In order to perceive the space, the brain exploits some attributes or cues to derive an estimation of the environment around the observer. These cues are in different types available for binocular and monocular vision, as well as for three-dimensional space or two-dimensional image of the space. For instance, when we move around the pattern of motion on our retina will be changing and therefore, we can obtain information about the spatial layout of the environment. This visual image displacements on the retina ("optic flow"), can help even small fast-moving animals, such as many insects, that are limited in binocular depth vision to very short ranges to derive depth information (Collett, and Harkness (1982)). Since the focus of this thesis lies on the visual space perception, in the following a number of the most important depth attributes or cues will be discussed.

Depth perception

Perception of environments is necessary to survival regards to self-protection, hunting as well as in concept of personal space. The perception of space is analyzed with how object's physical appearance is recognized, or its interactions are perceived such as visual space. To do so we need to perceive the environment in three dimensions and distances to the objects which is called depth perception. Regardless of what geometry best describes our perceptual experience, we need to combine information from the multiple cues available in our sensory input to perceive the object shape and its distance. For instance, it has been shown that in (Clark and Yullie 1990), the visual system, by combining information from several depth cues, can estimate 3D layout with greater precision across a wider variety of viewing conditions than it could by relying on any one cue alone. In the following section a brief overview of the research on visual distance perception will be provided and avoid having an exhaustive review here and refer to the Creem-Regehr and Kunz (2010), Cutting and Vishton (1995), Loomis and Philbeck (2008), or Proffitt and Caudek (2002) for a more thorough review on distance perception.

Pictorial cues

When we consider the depth cues that our brain uses to perceive the depth based on the interpretations of information derived from a motionless scene such as image, we are talking about the cues that Goldstein (2007) called them pictorial cues. These cues explain how we perceive the three-dimensional space around us by using an upside-down two-dimensional image on the retina. Cutting, and Vishton (1995) listed 'pictorial occlusion', 'relative size,' 'linear-perspective,' 'aerial perspective,' 'height in the visual field, as the pictorial cues, however some other introductory perception texts (e.g., Bruce, Green, and Georgeson, 2003) added more cues to this list.

Pictorial occlusion or interposition is the cue when objects further in the background are occluded by those closer to the observer (Figure 2), a topological alignment of depth will be created by perceiving the occluded object further away than the occluding object. Although the pictorial occlusion or overlap only indicates relative, and not absolute distance, it is still effective over the whole range of perceivable distances. Relative distance refers to the tendency to visually perceive the object that produces the larger image on the retina as being closer, and the object that produces the smaller portion of the object's surface is in the field of vision will be estimated as the smaller object in the scene.

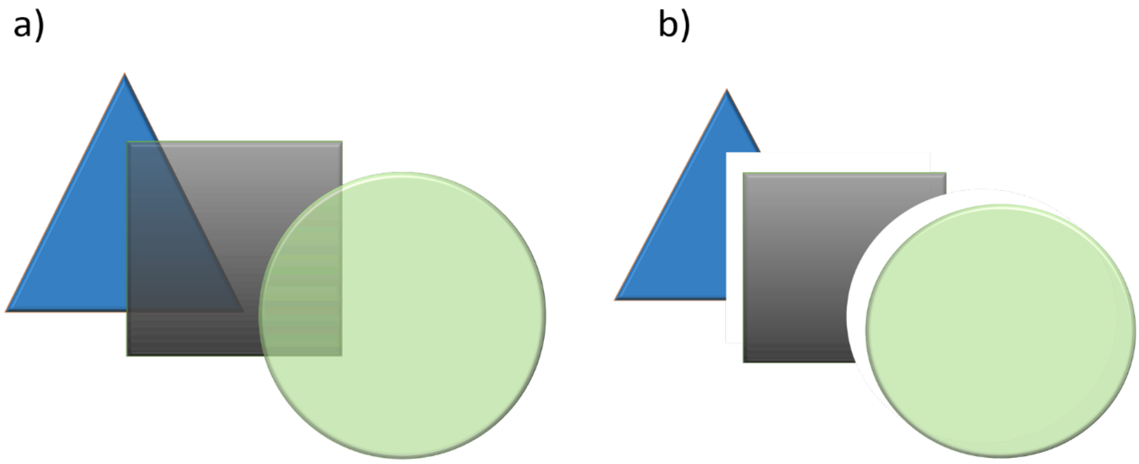


Figure 2 Occlusion

Linear perspective describes that parallel lines in space converge toward a common vanishing point, closely related to texture gradient which refers to the gradual reduction of detail that occurs in a surface as it recedes into the distance, compared with a surface that is close and perceived in fine detail (in Figure 3 both circles have the same size).

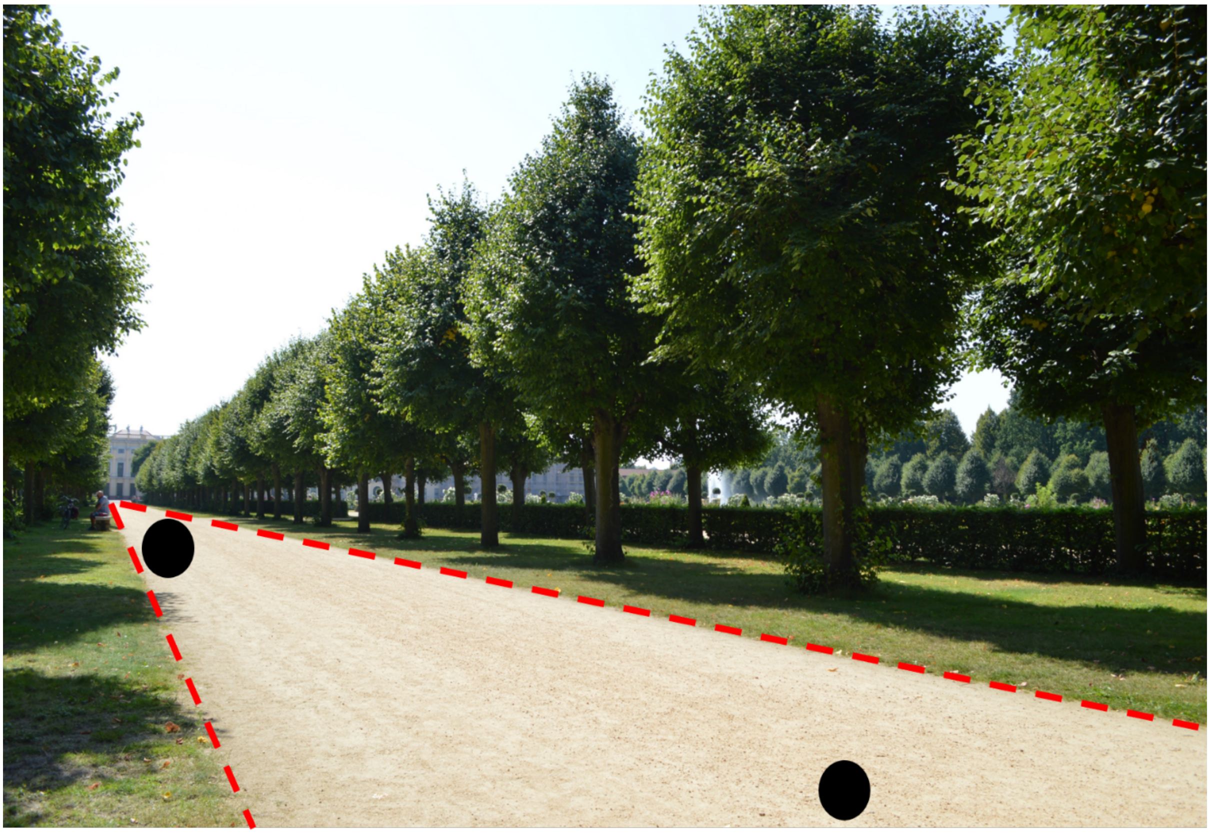


Figure 3 Linear perspective and texture gradient

Height in the field of view also called horizon-distance relation or angle of declination, suggests that the observer perceives the objects located higher up in the field of vision to be further away when both observer and object stand on a ground plane. To be able to account the height in the visual field as a valid cue to distance the presumptions that specify that the objects are resting on a single, reasonably flat plane, and that the plane is below the point of view of the observer have to be fulfilled. These conditions are usually violated in a typical indoor environment in which objects are placed in arbitrary surfaces including multiple horizontal and vertical surfaces at different elevations, such as floors, walls, and shelves. Aerial or atmospheric perspective describes that objects in the distance are bluer and decreased in contrast owing to particles in the air ("fogging effect"). This depth cue provides only relative distance information, and its effective range varies with weather conditions.
As it was mentioned many other depth cues also have been listed as the pictorial cues. For instance, "Familiar size" which refers to the condition when the observer knows from experience that objects become smaller when they are located in further distance, and shows the observer can draw corresponding conclusions concerning the unknown parameter If either the object size or the distance is known. In addition, shadow cues, relative brightness, and many other items such as relative density which describes that clusters of objects have a higher retinal density when they are farther away have also been categorized as the monocular depth cue. To avoid having an exhaustive list in this introductory section, we do not list and describe all the other pictorial distance cues that have been identified and discussed, sometimes in slightly different forms by different authors.

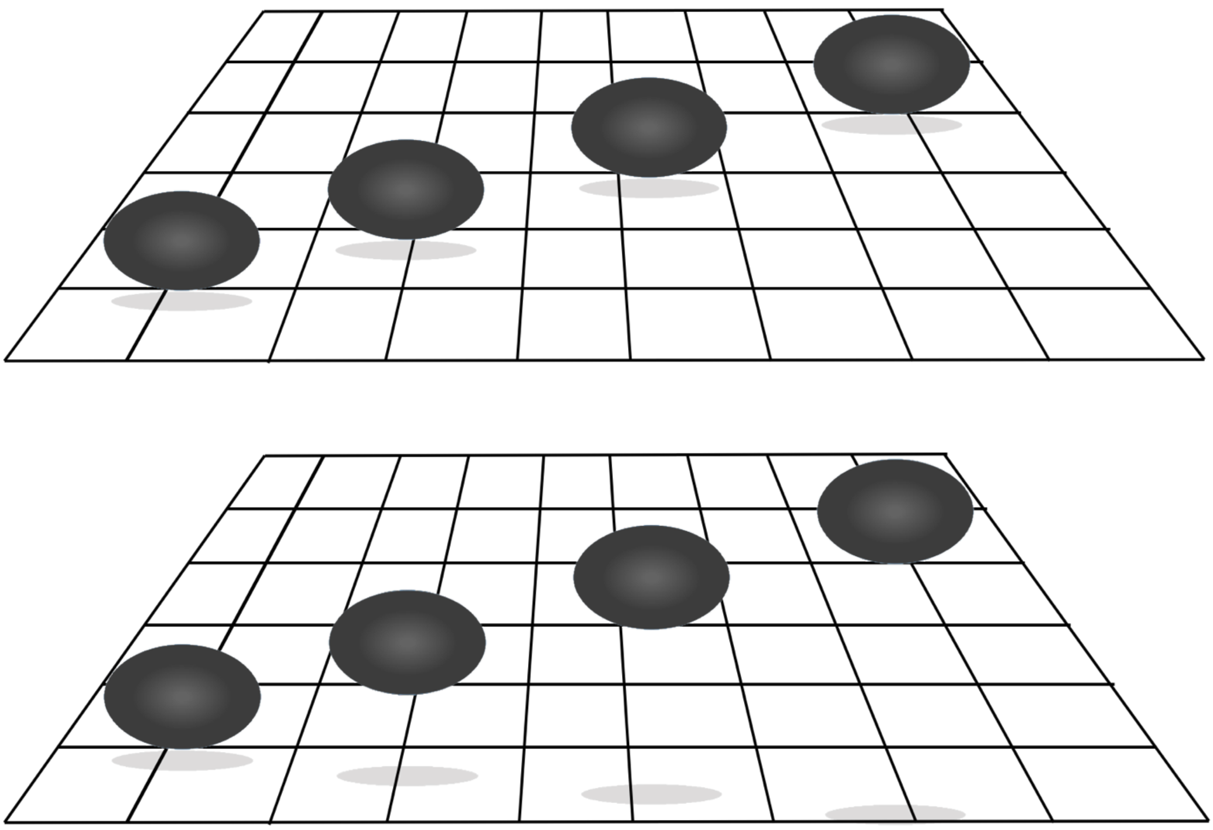


Figure 4 Depth from the shadows

Nonpictorial cues

As opposed to pictorial depth cues, nonpictorial depth cues are not available, or at least not valid, in two-dimensional, and motionless images. There are in fact additional sources of distance information that generally providing more precise depth information than pictorial cues (Nadler et.al., 2013). This information could be derived either from motion (motion parallax), the oculomotor system (convergence and accommodation), or the fact that humans have two eyes (binocular disparity).

Since each eye views the environment from a different vantage point, the images on each retina differ slightly. The resulting small disparities between the two monocular images constitute visually important information not available in either image alone (see Figure 5).

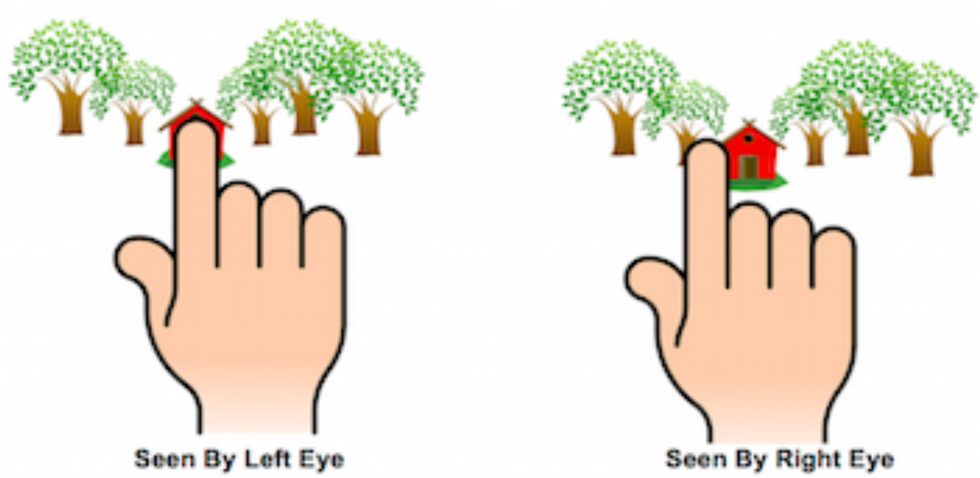


Figure 5 Binocular Disparity (http://hotgurlz445.blogspot.com/2013/11/to-better-explain-retinal-disparity-i.html)

Binocular disparity refers to the difference in position between the two retinal image projections of a point in 3D space (Marr and Poggio 1976, 1979). Binocular disparity is the strongest depth cue that makes the robust percepts of depth obtained exploiting binocular vision (Lappin 2015; Cutting and Vishton 1995; Proffitt and Caudek 2002).

In 1971 Julez demonstrated in his elegant experiments with random-dot stereograms that the brain can compute depth from binocular disparity cues alone. These random-dot stereograms (RDS) contain large numbers of identical elements with countless potential binocular correspondences and disparities. He demonstrated that for the observer viewing RDS (Figure 6) objects appearing to be in front of or behind the display level in the absence of any cues available to either eye alone.

Figure 6 A classical random-dot stereogram (RDS).

Motion parallax due to translational movements of the observer or the observer's head provides information about the distances to surrounding objects. When an observer moves, the retinal images of stationary objects off to the side of the direction of movement move as well. For closer objects, a given amount of translation of the point of view will cause a larger change in angular location than for more distant objects. This motion parallax isolated from binocular and pictorial depth cues, motion parallax can also provide precise depth perception and it is an important depth cue for shorter distances, but its effectiveness declines with distance [Cutting and Vishton 1995].

The change in the curvature of the lens of the eye allowing it to focus on objects at various distances is called accommodation. In order to keep the image of an object sharply focused on the retina, the lens of the eye must change shape slightly depending on the distance to the object. In consequence the objects at other distances, either nearer or further away, will appear blurred. As the eyes move inward to look at nearby objects, the angle between the optical axes of the eyes decreases, which is called convergence. Since the visual system obtains information from the muscles that control convergence and accommodation, they are considered as depth cues.

Cutting and Vishton [1995] divided the environment into three circular regions around the observer, called personal space, action space, and vista space (see Figure 7) to be able to depict the different effective ranges of the aforementioned depth cues.

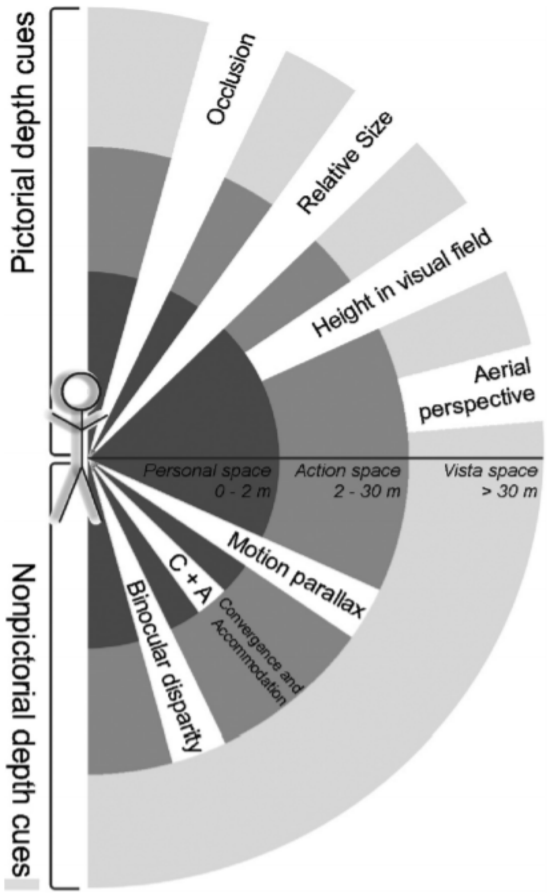


Figure 7 Schema of effective ranges of the different depth cues (based on Cutting and Vishton [1995])

The influence of context and personal variable

Apart from depth cues, it has been recently suggested that distance perception might not only be influenced by the availability and reliability of depth attributes but also by environmental context and personal variables. For example, Lappin et al. in 2006 demonstrated that the accuracy of distance is subject to the subtle properties of the surrounding environment and not always accurate even in the real world (Lappin et al., 2006). In addition, Witt et al., found effects of environmental context for both egocentric and exocentric distances. Their results indicated that the space beyond the target, with no relevant depth cues, can influence perceived distance. They concluded that distance perception can be influenced by environmental context (Witt et al., 2007)
In addition to the enviomental context effects of the perception of the distance several recent studies show that personal variables of the observer, such as the physiological state or the intention to act could play a role on observer's perceiving of the spatial layout of their environment. For instance, researchers have shown in a series of experiment that perceptual experiences of spatial properties like distance, slant, and size can be differed by manipulating the effort associated with walking influenced perceived distance only when observers intended to walk the distance. (e.g., Bhalla and Proffitt, 1999).

In another study, objects in personal space appeared closer when a tool was held and thus the object became reachable (Witt and Proffitt. 2005). This effect happened only when the observer intended to reach for the object. In a review, Proffitt concludes that perception "is influenced by three factors: the visually specified environment, the body, and the purpose" and is therefore action specific (Proffitt et al., 2008). The idea that ability to act has an influence over one's perception is known as the action specific account of perception (Proffitt, 2006; Witt, 2011, 2017).

Focus of the Thesis

As it was discussed briefly the physical space is different from the one that we visually perceive. This visually perceived space surrounding us is three-dimensional, but its intrinsic geometry is not Euclidean. We reported that many studies showed that perceptual geodesics deviate significantly from the true straight lines of Euclidean space.
In this thesis work, we repeated first the exocentric pointing experiment by Koenderink et al. (2000, 2003) who measured tangents to the geodesic in the horizontal plane at eye-level and then extended to the elevated levels. The details of these behavioural experiments are reported in the next two chapters. In chapter four we demonstrate the results as well as the data analysis to state the findings of our research. In the discussion chapter we evaluate our findings and compare our results with mentioned papers and finally conclude how our research work relates to the literature.

Chapter II: Methods

The previous chapter discussed the difference between visually perceived space and actual physical space and further showed that how historically the interpretation of the geometry of the visual field has been evolved. Euclidean and non-Euclidean geometries and their differences were also presented to provide a brief review over the most important and relevant topics needed to comprehend this research. We also explained the most relevant and recent literature and ultimately highlighted the main motivation of this research. In this chapter, the research methodology explains how the proposed study was fulfilled. We define the methods that we used to collect data and illustrate how we configured the experiments.

To investigate exocentric visual direction toward targets, the exocentric pointing task was implemented in these experiments. The following simple, and direct method (see Figure 8) to measure angles in visual space was originally created by Koenderink and his collogues in 2000 and improved later in 2003 (Koendering et al., 2000, 2003). In contrast to the original experiment that was performed in outside environment, we carried out the experiments inside with two conditions of block (wall) and unblocked (open). In addition, in our third experiment we improved the original concept by performing an exocentric pointing task at elevated eye level and collected a set of novel data comparing to Koendering's research method which only considered the task at eye height level.

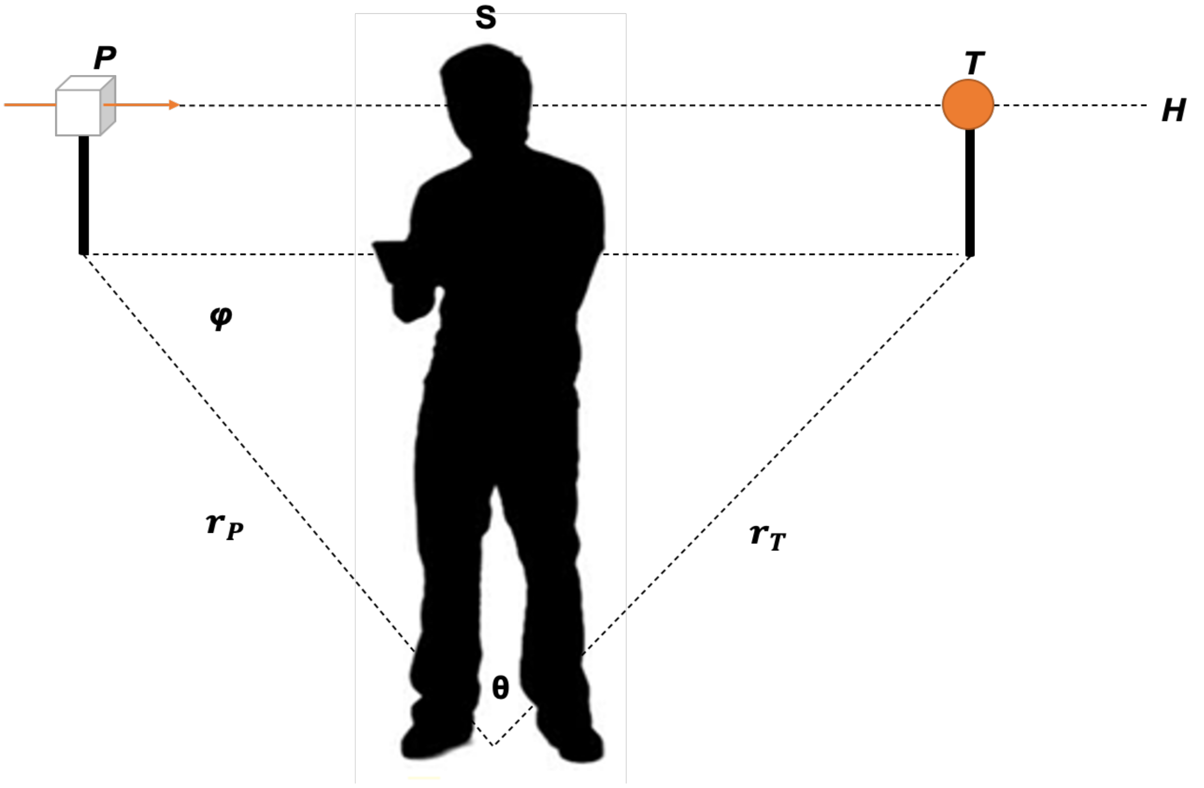


Figure 8: The exocentric pointing paradigm. The subject standing upright S, target T and pointer P are at the same vertical height. Thus, the pointer and target appear at the horizon of the observer. The relevant geometry

has been drawn on the ground plane. Pointer and target appear at an angle of h from each other on the horizon. The veridical pointing angle is $\boldsymbol{\varphi}$. The pointing angle is even monocularly evident from the appearance of the pointer. The veridical pointing angle phi depends on the angle h and on the ratio of the distances from the observer to the pointer (r_P) and the target (r_T).

A pan and tilt remotely controlled pointer were mounted on an adjustable in height tripod. The pointer was a paper cube, colored in white and in size of 22 cm edge length, pierced with a wooden arrow in thickness of 1.5 cm and total length 100 cm. The arrow's head was in 3 cm in diameter as well as 1.5 cm in long and was painted in shiny, bright orange color. The arrow protrudes out at both sides. The pointer also was included a very light video camera that was connected to a laptop which placed next to the pointer, to take pictures from pointing experiment and collect them on to the laptop for further analyzation. To control orientation of pointer we used a pan and tilt motorized device, controlled remotely with a relevant app on a mobile phone (seeFigure 9 and Figure 10).

Figure 9 Experiment setup

Targets were different in size and bright orange spheres mounted on adjustable in height tripods. Spheres diameters were adjusted to distances from pointer and subject and they were 16-24 cm. Subjects apparently were different in height. Thus, the heights of both pointer and targets were adjusted to the eye height of each observer. Since the pointer was at a fixed

distance and the targets were adjusted in size, retinal image size could not be an important cue. Locations are accurate to 10cm, pointer orientations to about 0.5°. This turned out to be sufficient given the repeatability of the observer's settings. (repeat the experiment 5 times in each position and left and right). The observer was instructed to keep the feet within a 40 cm diameter area and to always stand upright. The observer was permitted eye movements, head rotations and torsions at the waist, even changes in placement of the feet (turning in place).
The explicit instruction of the experiment was to control the arrow and ultimately point it towards the targets in different distances from pointer and subjects. Since the subjects were not at the pointer to point exactly to the targets (exocentric pointing), obviously this was a trivial task to exactly point where the targets are located. Observers were not informed of their settings at all time and most of them found the task a quit ''natural'' one. A mobile phone was given to subjects at the beginning of the experiments as a remote control for the pointer and to adjust the pointer direction towards the position of the pointer as their best guess. The experimenter captured a picture as soon as subject adjusted the position of the pointer and concluded the closest estimation, by the camera that was mounted to the box of pointer. The camera was placed exactly under the arrow so that the head of arrow and the target could be shown in each picture. Consequently, pictures were collected via a laptop connected to the pointer camera automatically for further analyzation. Measuring geodesics and all the relevant analysis was performed in Matlab and will be discussed in the next chapter.

Figure 10. Pointer and target at the same picture

Chapter III: Experiments

Experiment 1

The framework of this straightforward experiment consists of a series of target positions laying on the same straight line. The length of the line on which the targets are placed is 22 meters. The subject was standing on a perfectly marked position while the pointer and the targets are positioned on the opposite side of the observer. As it can be seen in the Figure 12 , the distance between the pointer and the target always exceeded 90° of visual angle. However, the position of the pointer and the observer always was constant at 2 meters distance. The reason why the latter was constant and the former was variable was the fact that the veridical pointing angle depends critically upon the ratio of these distances (see

Equation 1). The veridical pointing angle φ depends on the arc between the pointer and the target (angle ϴ) and on the ratio of their distances r_p over r_T . In other words, the positions of the targets and their sizes were the only variables that changed during this experiment and therefore all observers found it necessary to make body and head turns in order to look back and forth between them.

Equation 1: $tan\,\varphi = \frac{sin\,\theta}{\frac{r_P}{r_T} - cos\,\theta}$

The essential pointing geometry is shown in Figure 11. Here the observer at S points the pointer P at the target T.

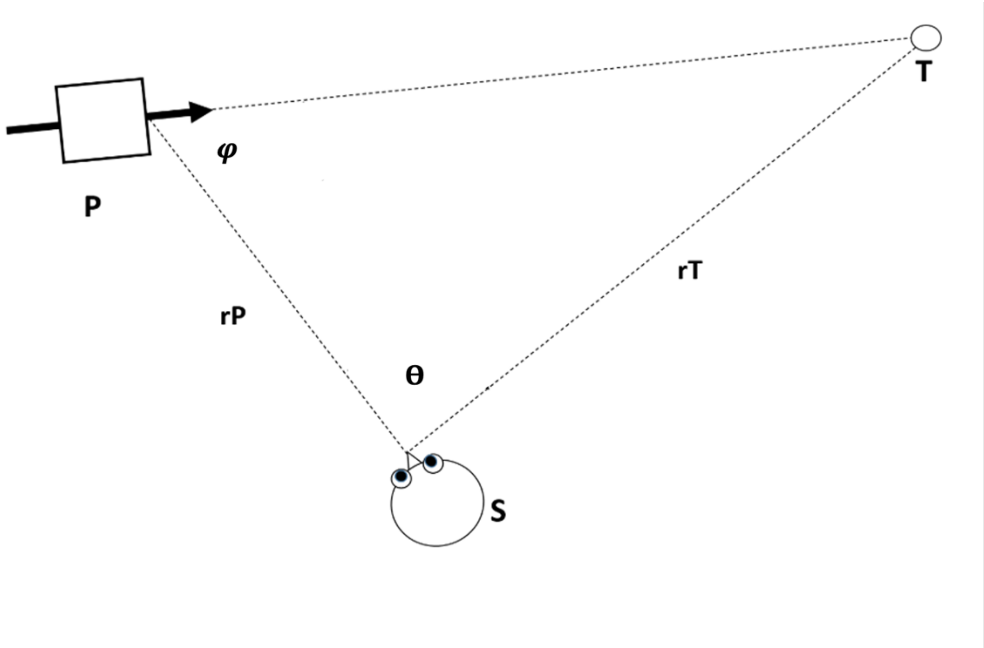


Figure 11: The pointing geometry. Observer at S, pointer at P, and target at T are in the horizontal plane at eye height. Thus, the arc # is a distance along the horizon. The veridical pointing angle j depends on the ratio of distances and this arc.

Targets were placed at five predefined spots on the linear track, with a visual angle of 150˚, 135˚, 120˚, 105˚, and 95˚. The size of targets varied base upon their distance from observer and the pointer. Three different sizes of targets namely small, medium, and large, were used respectively in different distances ranging from near to far. An average of a symmetrical target positions (towards left and the right of the observer) were used in this experiment. There was a similar value in random samples from complete series, thus we have no reason to expect artefacts due to order. Besides, subjects were asked to point to himself / herself, an absolute calibration of pointer orientation, thus increasing. All five target's places were visited in increasing order once per session. A total of ten sessions were conducted for each subject in this experiment meaning that the total number of pointing per target was ten. In order to avoid overconfidence bias, subjects pointing task was performed for all targets at each session instead of repeating for each target ten times so that subjects could not refer to his / her memory to adjust the pointer.

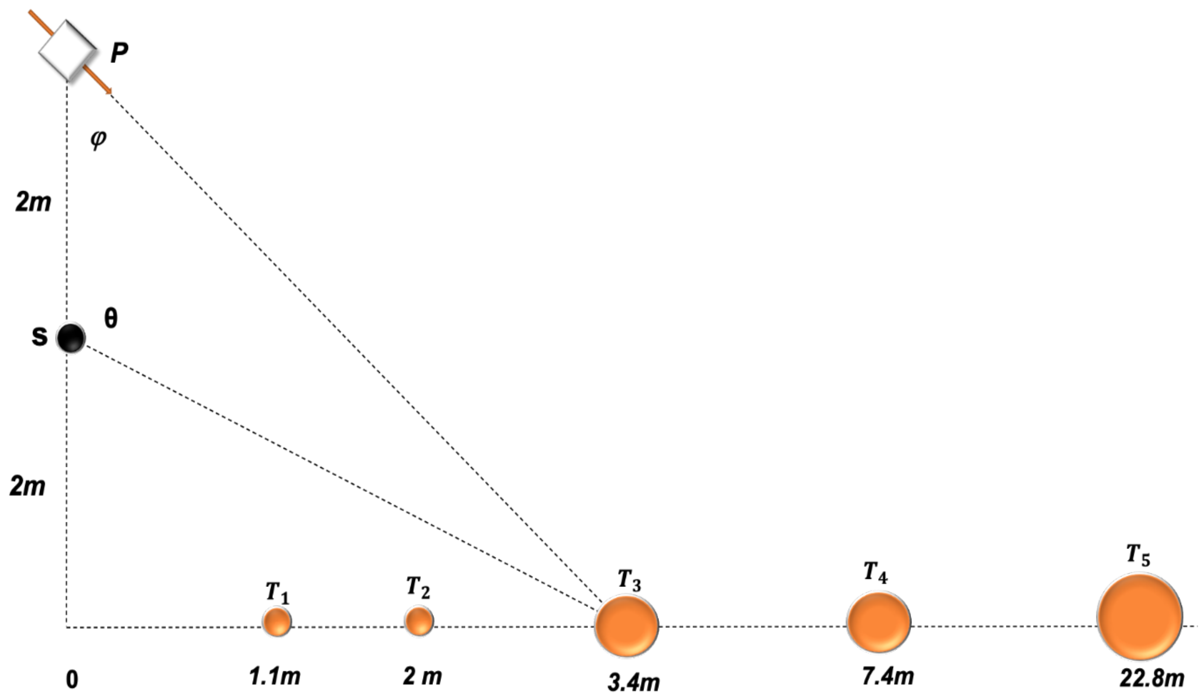


Figure 12 Experiment 1 set up

In this experiment we ran the experiments with two conditions called “open space” versus “wall”. To observe that if there are cues that could help subjects for the pointing tasks, we tested the same experiment first when there was a wall behind the targets in comparison with no wall case. The place where the test was done had a good lightening condition, so there was no forced due to time constrains such as doing the experiment within a day or in sunny day , but a little time consuming because of moving the targets during and after each round of the test .

the angle φ is defined as the angle of pointing (taken relative to the visual direction towards the pointer), the angle ϴ is defined the visual distance between the target and the pointer at the horizon plane. r_P and r_T are the distances between subject to the pointer and the target, respectively. The important point is that the pointing angle is only based on these two ratios. It was previously shown in (Koenderink et al., 2003) that various monocular and binocular cues are available to estimate such ratios, although introspectively this is not what the observers do. The pointing angle φ can be judged through both binocular and monocular cues and further demonstrated that an error analysis suggests that monocular cues may well dominate.

Experiment 2

As it was discussed in the introduction chapter, Koenderink et al., (2000) probed human optical space, by introducing a novel method called exocentric pointing task which was repeated qualitatively in the first experiment. In this research paper, they addressed the Luneburg hypothesis that human optical space should be of constant (negative) curvature. According to their data this was clearly falsified meaning that the horizontal plane at eye height has an elliptic curvature in the near zone and a hyperbolic curvature in the far one. This finding was an explicit conclusion from this paper and was improved later in Koenderink et al., (2003).
In experiment 2 however, we tested an implicit prediction from the Koenderink et al. (2000) research paper. They predicted that the observer would need to choose between two equally 'visually correct' exocentric pointing directions. In other words, the geodesics from pointer to target are modeled as circular arcs to which the pointer is tangent. In a symmetric arrangement as depicted in Figure 14, this implies that the pointing direction in the midline of the arrangement should always be frontoparallel to the observer.

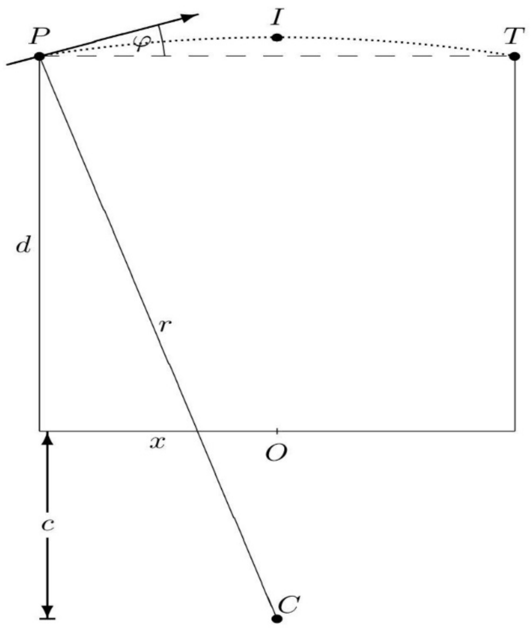


Figure 13 Intermediate points on geodesic

We tested this by first measuring the pointing error ***φ*** for a given pointer and target position. We carried out this experiment inside the same building with the same subjects. In our pre-experiment phase, two different distances (4m, 6m) between target and the pointer were selected. Observer was standing exocentrically, at two meters distance from horizontal plane where the pointer and the target are positioned at the same line in front of the subject. We always used symmetrical positions (target to the left and the right of the observer) to test and measure pointing error ***φ*** for a given pointer and target position and finally calculated the average value. Initially, it was assumed that this angle could be approximately equal when the pointer was to the left or right of the observer. However, after performing the experiment, we found that not only was it always not equal, but also sometimes its sings were different which will be discussed in detail in the next chapter.
We then estimated the "intermediate point" ***I*** where exactly the assumed geodesic intersects the observer's midsagittal plane. We then placed the pointer at this intermediate point and

repeated the measurement. The point ***I*** was then calculated as follows. The position of the observer is denoted by ***O***, while ***P*** refers to the pointer, and ***T*** signifies the target. The left-right-distance between the observer to the pointer as well as to the target is marked as ***x***, and the distance from the baseline to ***P*** and ***T*** is shown by ***d***. We calculated the radius of the circular geodesic by drawing a straight line from ***P***, at right angle to the pointer, into the sagittal midline of the configuration. Its endpoint is the sought center of the circle ***C*** and the length of this line is its radius ***r***. Observing that the angle between this radius and the sagittal direction is again ***φ***, we have:

$$d + c = \frac{x}{\tan \varphi}$$

and

$$r = \frac{x}{sin\varphi}$$

From this, the distance from the observer to the intermediate point can be calculated as

$$\overline{OI} = r - c = d + \frac{x}{sin\varphi} - \frac{x}{\tan \varphi}$$

which can be shown to reduce to

$$\overline{OI} = d + x\, tan\frac{\varphi}{2}\ .$$

The formula holds also if the pointer is pointing inwards if we count such angles as negative.
Having calculated the intermediate point ***I***, which were different between subjects, we considered four different points around point ***I,*** two of them close to the subject and two further away and placed the pointer at each point at every block of the experiment (see. Figure 14). The position of the target was constant and at eye level height of the subject. We ran this experiment in five blocks and then averaged over the five repetitions at each point on which the pointer was located (Figure 14).

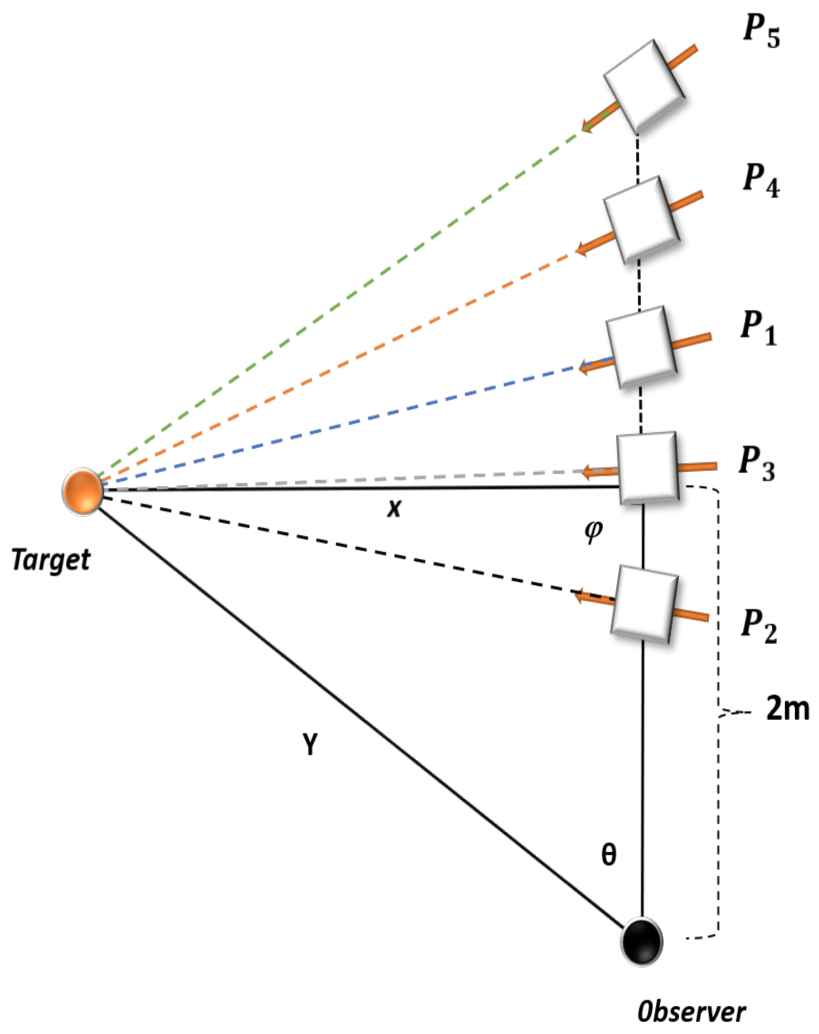


Figure 14 Experiment 2 Setup

S = subject, P= pointer, I= intermediate point,

Experiment 3

As it has been shown in previous experiments, the original method presented by Koenderink et al., (2000, 2003) was essentially exocentric pointing task in which tangents to the geodesic in the horizontal plane at eye-level was measured. In this experiment, however we measured the tangent to the geodesics at upper eye level vertically and repeated the relevant details.
In order to conduct this new experiment, we had to modify the pointer structure as well as the camera position. Since the pointer structure used in the previous experiments was not sufficiently light to be able to be tilted vertically, we redesigned its structure this time using a very light wooden cube with an ultra-light wooden arrow pierced on it. The same remotely controlled pointer was again mounted on a tripod and the arrow stuck out at both sides. In addition, the cube and arrow were not painted but in neutral woods color and the pointer motor

was the same with previous experiments. Since we needed to have a vertical movement of the pointer, the camera was pierced to the side of the box near the arrow to be able to use the same calibration settings (Figure 15). All the rest of the details kept intact.

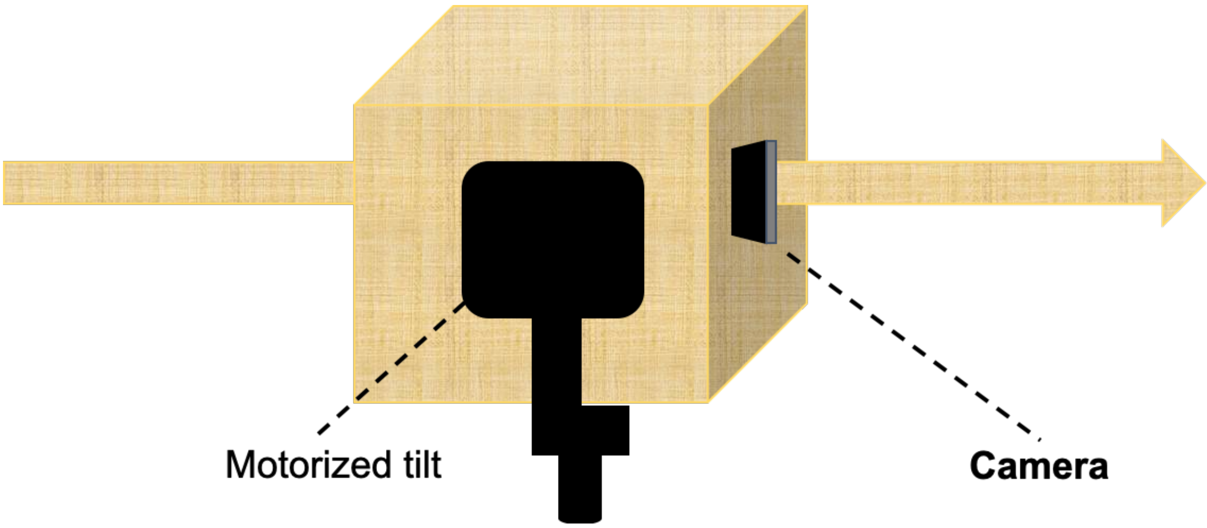


Figure 15 A new design of pointer and camera. location

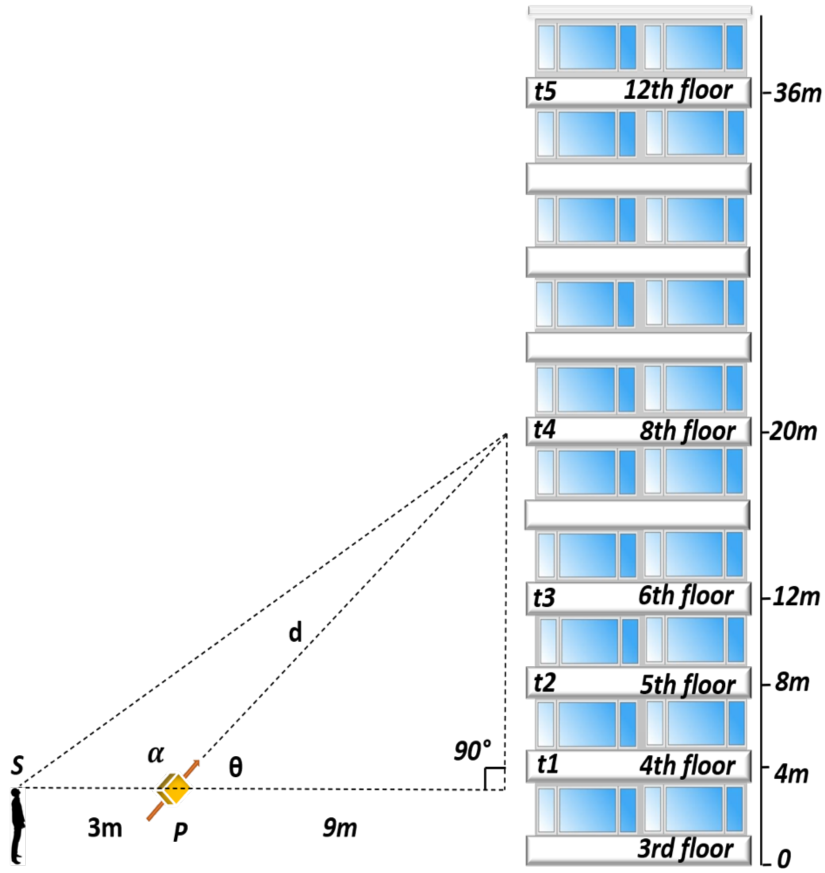


Figure 16 Measurement of geodesics at elevated levels

The number of subjects participated in this experiment was four. The experiment was done in a balcony of the third floor of a building at the university campus. As it can be seen in the Figure 16 targets were a visible line in the middle of the balconies of five floors namely 4th, 5th, 6th, and finally 12th of this building. The details of the experiment setup are shown in the Table 1.

Table 1: Experiment 3 setup numbers

Floor number	α (degree)	Θ (degree)	Hight from pointer to target (meter)
4th	167.36	12.64	2.04
5th	146.43	33.57	6.04
6th	132.19	47.81	10.04
8th	116.77	63.23	18.04
12th	104.97	75.03	34.04

Chapter IV: Results and Data Analysis

Result 1:

In the following plots the systematic pointing errors as a function of the visual distance of target and pointer have been illustrated. The data are expressed in terms of the angle subtended by the visual direction to the pointer and the exocentric pointing direction. We have calculated these errors both form left and right side as well as open ended and wall ended conditions. Subjects "SOA", and "MHA" were naïve to the experiment while "BSH" and "HJD" were non-naïve. In contrast to the results of the original experiment (2003) in which the systematic errors for all observers were quantitatively and qualitatively very similar, we did not find any similarities among our subjects' results regardless of the condition type. In the mentioned study authors reported that for medium distances (r_T) all observers exceed the veridical angle sometimes severely (10–15), for near and far targets they are close to veridical or even show signs of under pointing. Although the pointing directions were always different from the veridical angles, in our finding there is no similar pattern in the systematic errors.

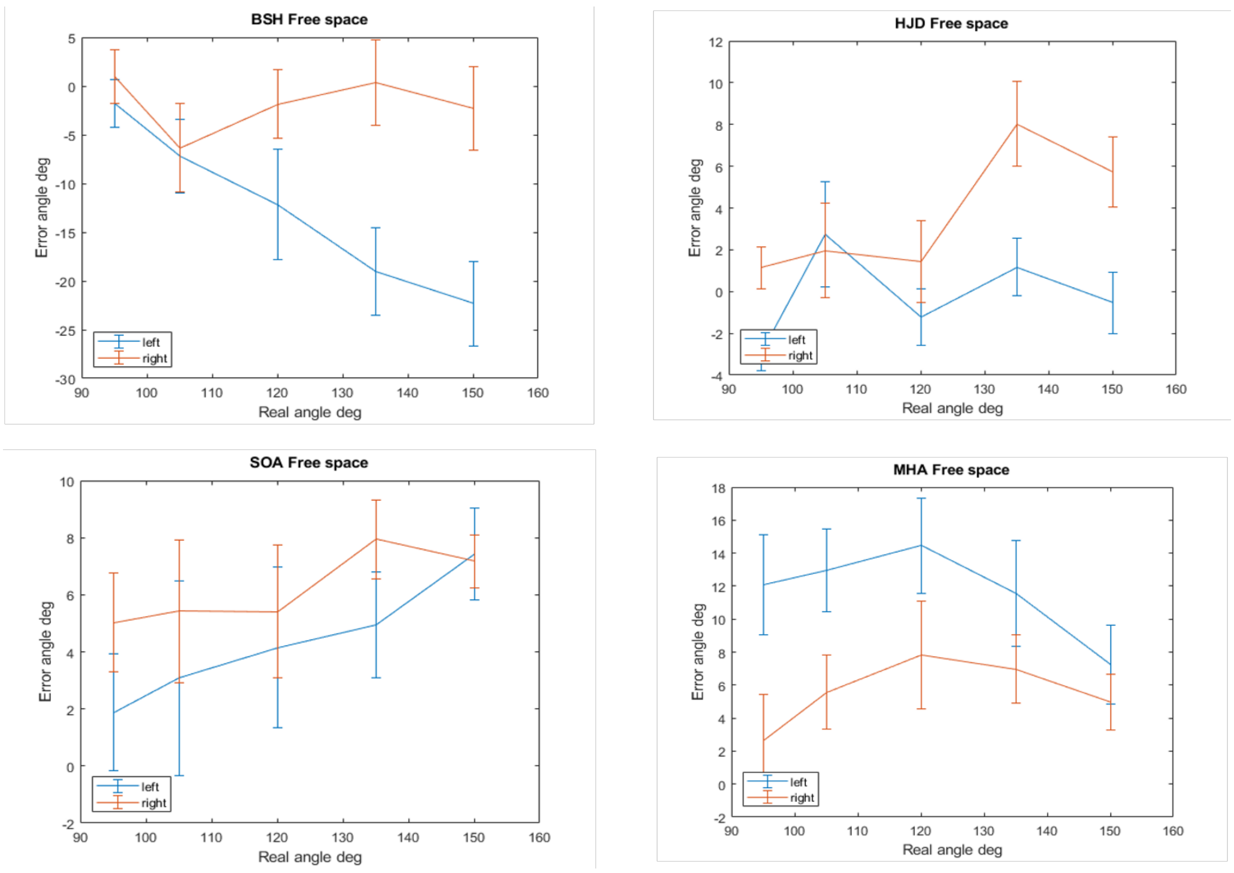


Figure 17 Systematic errors in free space condition.

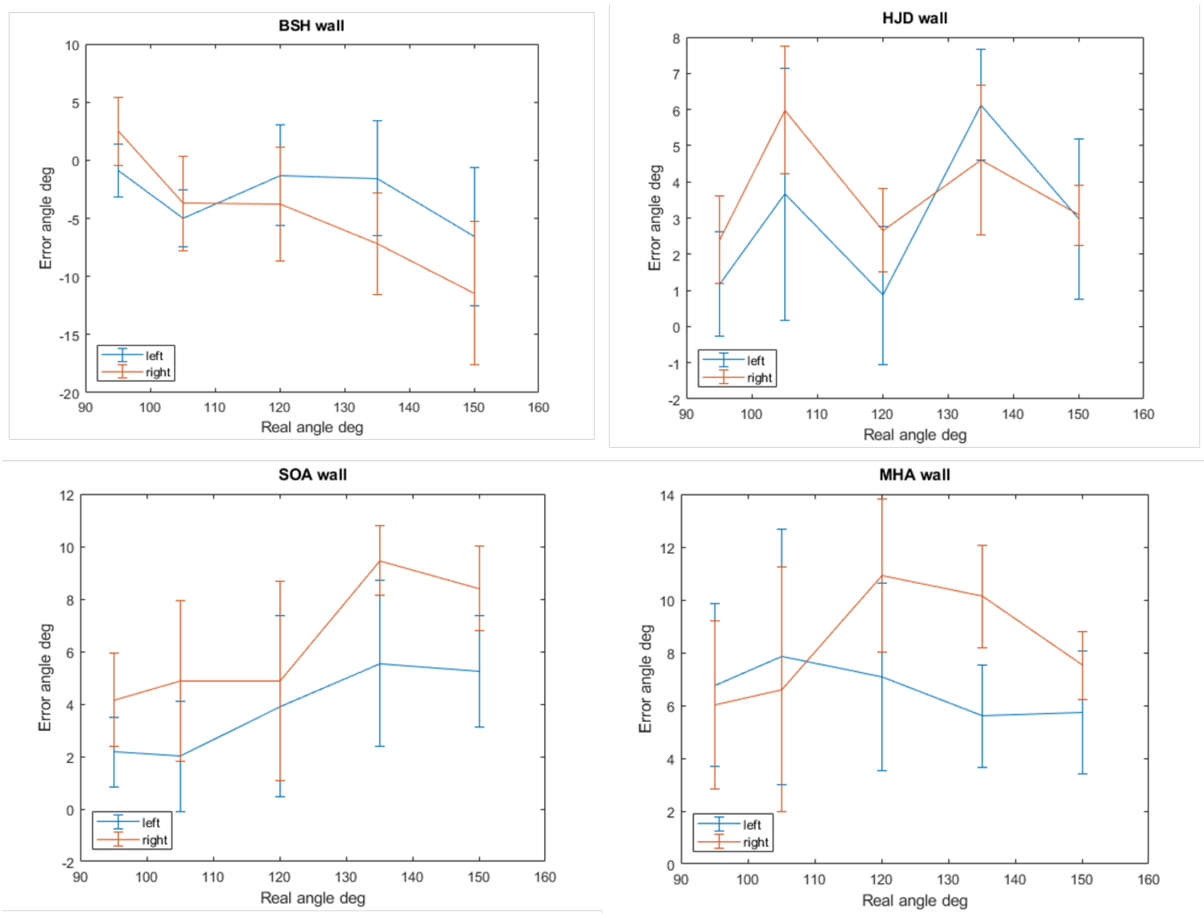
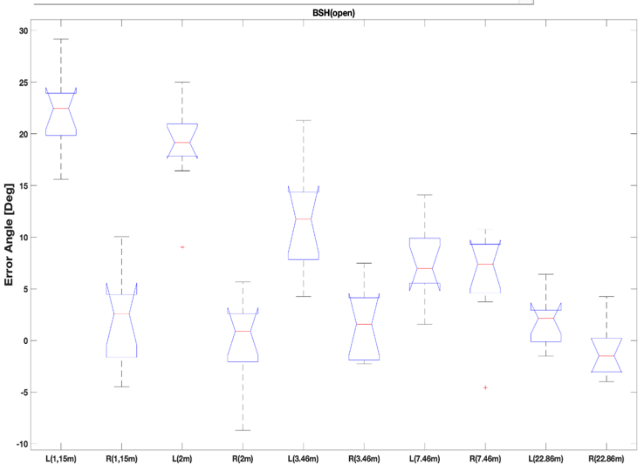


Figure 18 Systematic errors in wall condition.

Our results do not support the observation which has been made in (J. J. Koenderink 2003) paper. In fact we found that not only is there not any similarities among subjects in different conditions but also there is a significant level of difference between the systematic errors for each subject when they point to the targets located in theirs right side or toward left. The following figures show this difference for all subjects. As the following figures depict the systematic errors for each subject has different patterns in “wall” and “open” conditions suggesting that we cannot replicate the results of the original paper and we discuss the possible explanation for these discrepancy in the discussion chapter.

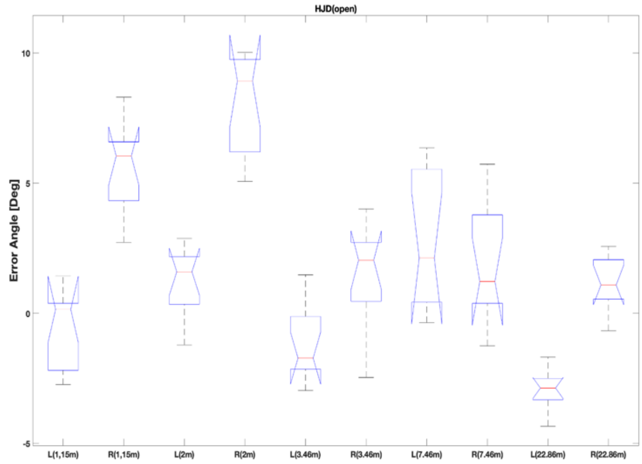
ANOVA Table

Source	SS	df	MS	F	Prob>F
Columns	6001.39	9	666.821	39.44	1.67779e-27
Error	1521.54	90	16.906		
Total	7522.93	99			



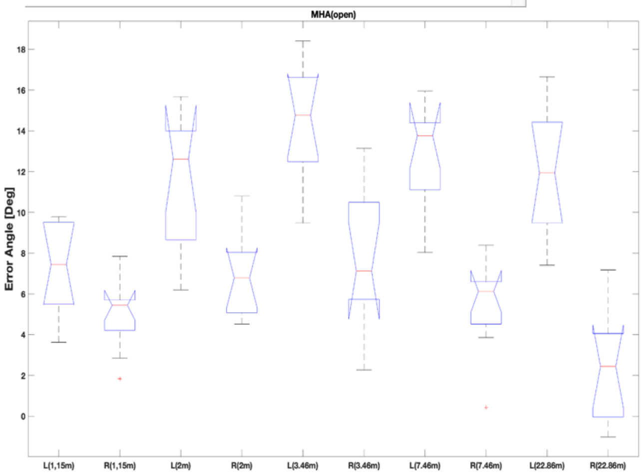
ANOVA Table

Source	SS	df	MS	F	Prob>F
Columns	936.11	9	104.013	34.98	1.08187e-25
Error	267.58	90	2.973		
Total	1203.7	99			



ANOVA Table

Source	SS	df	MS	F	Prob>F
Columns	1371.04	9	152.338	21.66	5.11264e-19
Error	632.92	90	7.032		
Total	2003.96	99			



ANOVA Table

Source	SS	df	MS	F	Prob>F
Columns	331.249	9	36.8054	7.81	2.00952e-08
Error	424.051	90	4.7117		
Total	755.3	99			

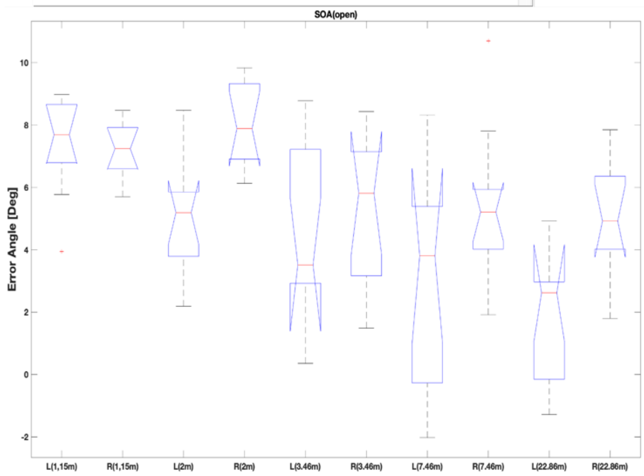


Figure 19 The box plot shows the difference between systematic errors of left and right side located targets in open condition

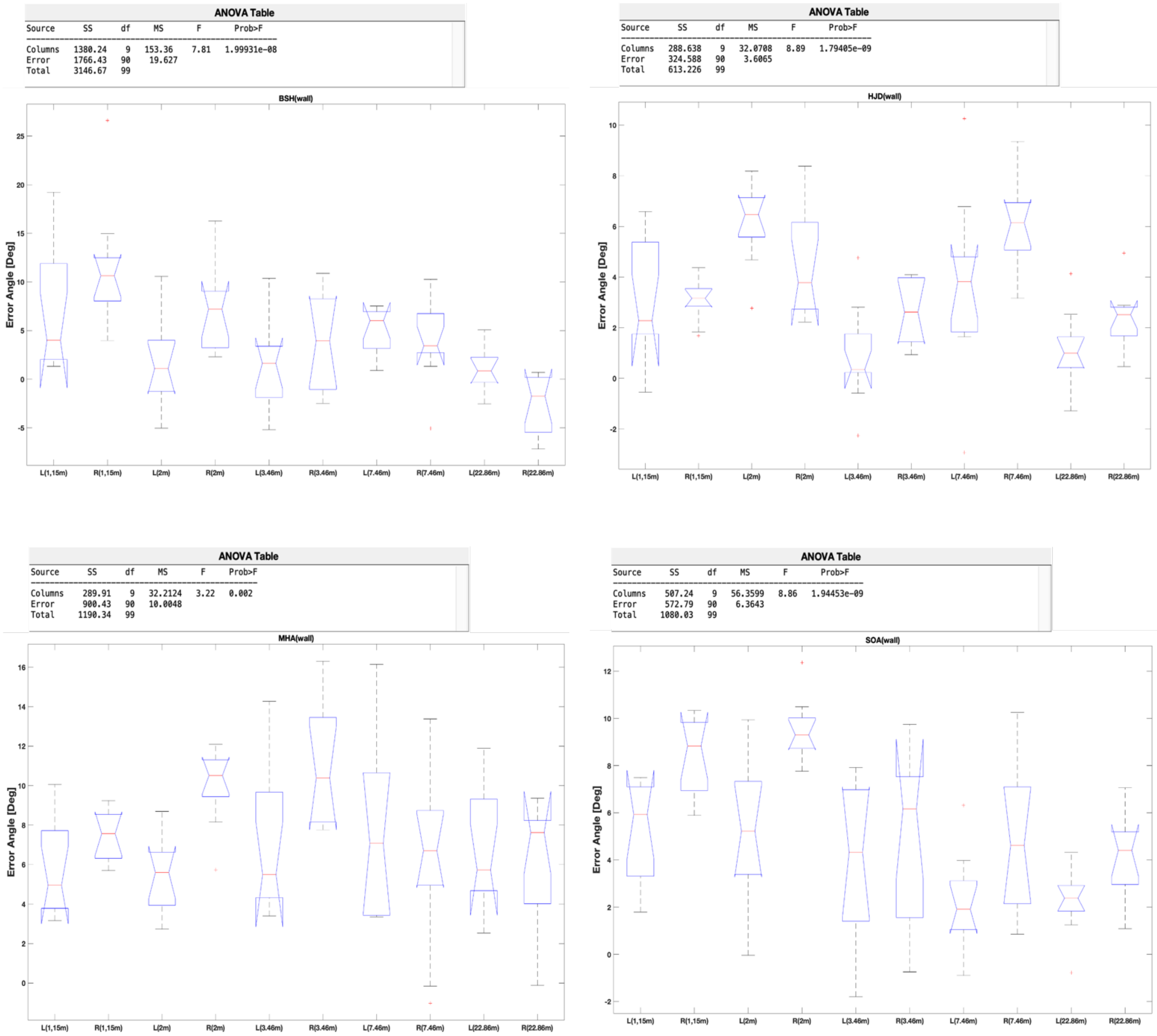


Figure 20 The box plot shows the difference between systematic errors of left and right side located targets in wall condition

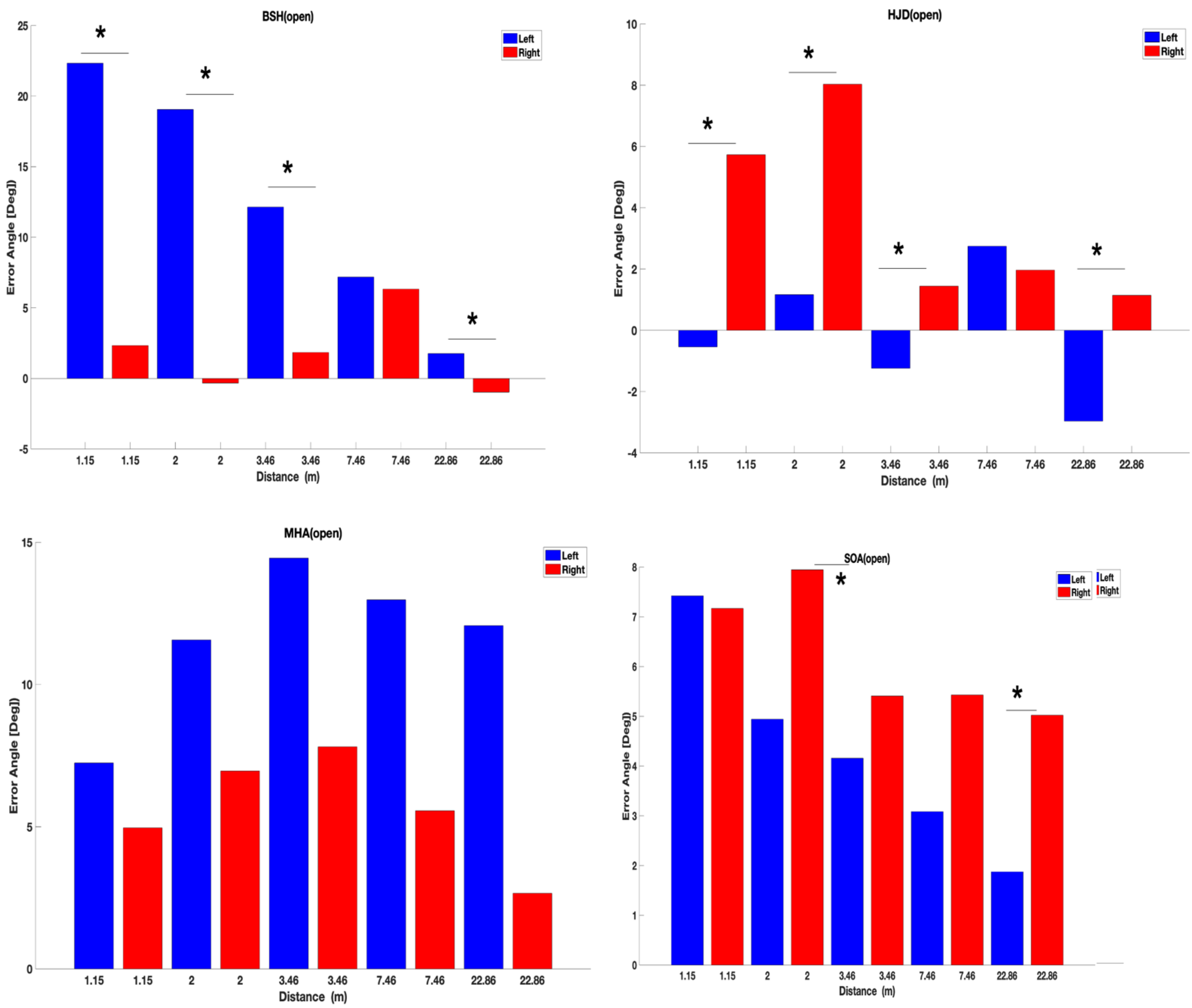


Figure 21 The bar plot shows the difference between systematic errors of left and right side located targets in open condition

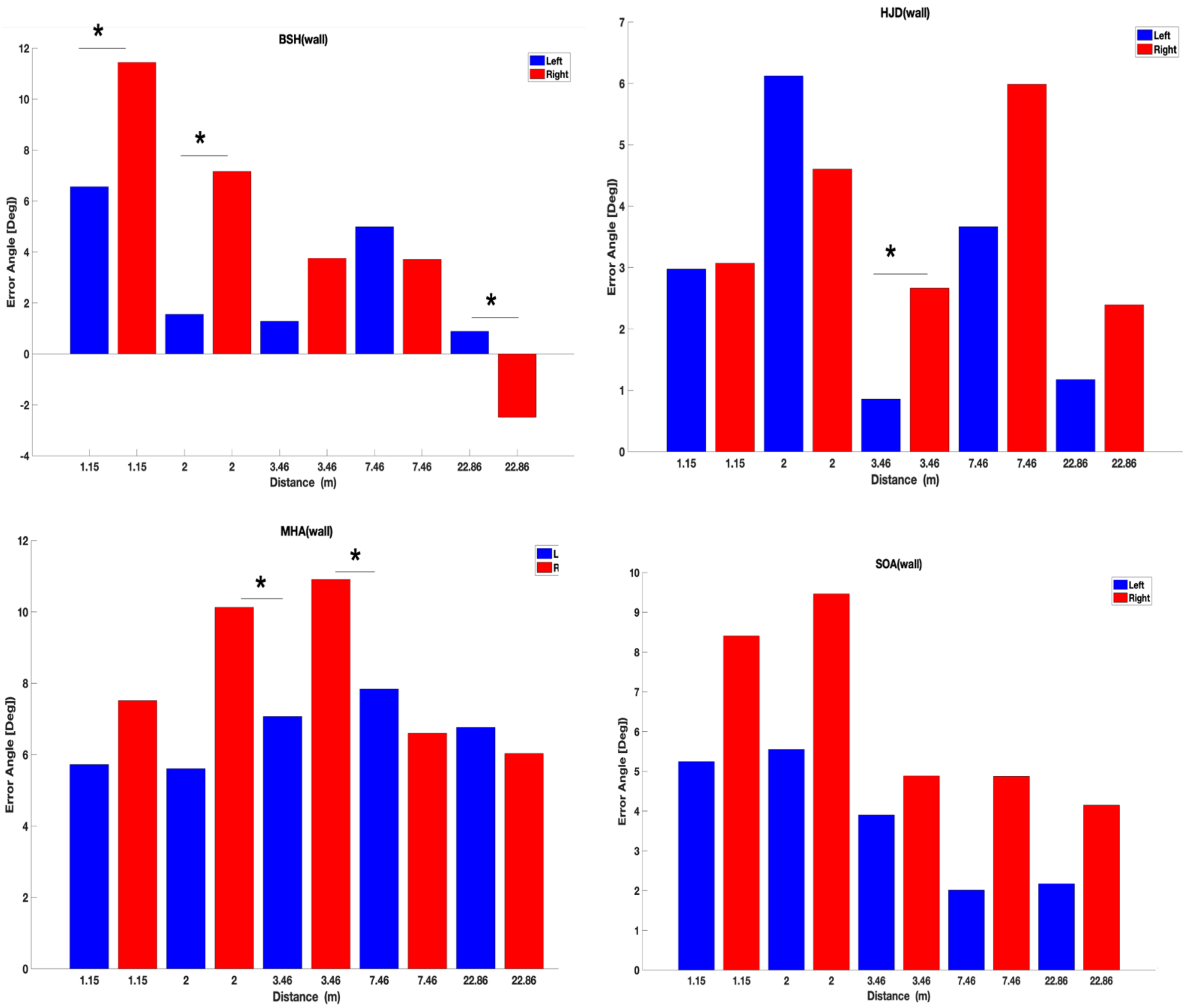


Figure 22 The bar plot shows the difference between systematic errors of left and right side located targets in wall condition

Result 2:

In contrast to the prediction that was made in Koendernik et al., (2003), our findings in the second experiment showed that the pointing angles φ_l and $-\varphi_r$ from the left and right sides differ substantially. This could mean that the geodesic is not a circular arc, in which case we cannot expect that the frontoparallel point (FPP) "I" is indeed in the midsagital plane. We calculated the two predictions for the deviation of I from the direct connection, which was calculated as Δd = x tan(φ/2). We then obtained two estimates of Δd, one for each pointing direction:

$$\Delta d_l = -x \tan\frac{\varphi_l}{2} \pm x \ \mathsf{sdv}\frac{\varphi_l}{2}$$
$$\Delta d_r = \quad x \tan\frac{\varphi_r}{2} \pm x \ \mathsf{sdv}\frac{\varphi_r}{2}.$$

Here we understand that φ_r is the angle shown in the figure, i.e., for rightward pointing from the left side. Positive φ_r lead backwards shift of FPP, i.e., positive Δd. The standard deviations are simply the standard deviations of the angular measurements.

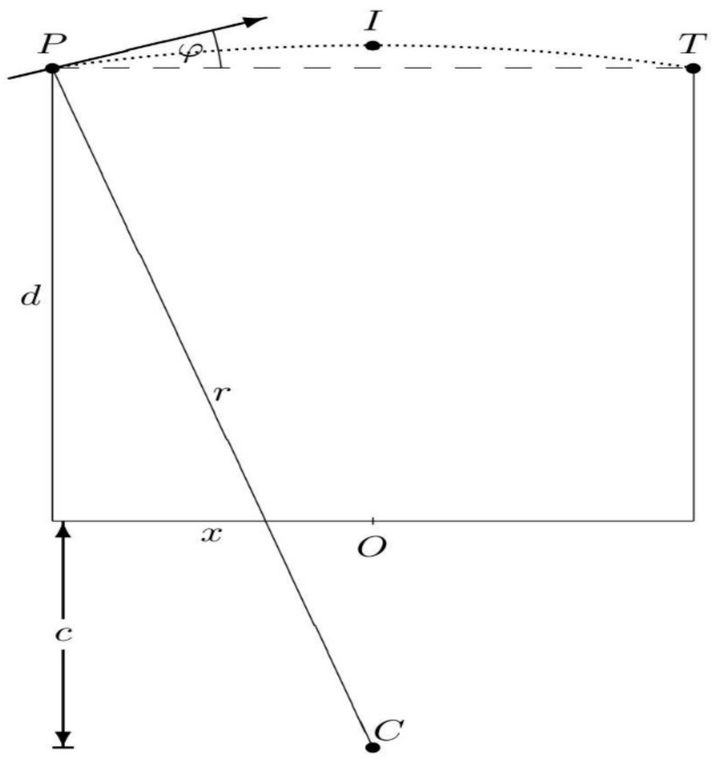


Figure 23 a Sample pointing tasks from the sagittal midline to the sides

Measurements of the frontoparallel point are obtained as a series of pointings from the sagittal midline to the sides (see.; Figure 24, Figure 25). We denote the positions of the pointer (distance from the Euclidean inter- section point $\Delta d = 0$) as p_i. For each position, we obtain measurements α_{il} pointing leftward (to the target in the right) and α_{ir}, respectively. Note that these are already the averages of the individual settings. Having analyzed the results of this experiment we realized that the rightward pointing setup was accidentally displaced for three of the subjects and only the subject HJD has been experimented with exact accordance to the theory for both sides. However, the remaining results clearly show the expected effect that we were looking for as follows.

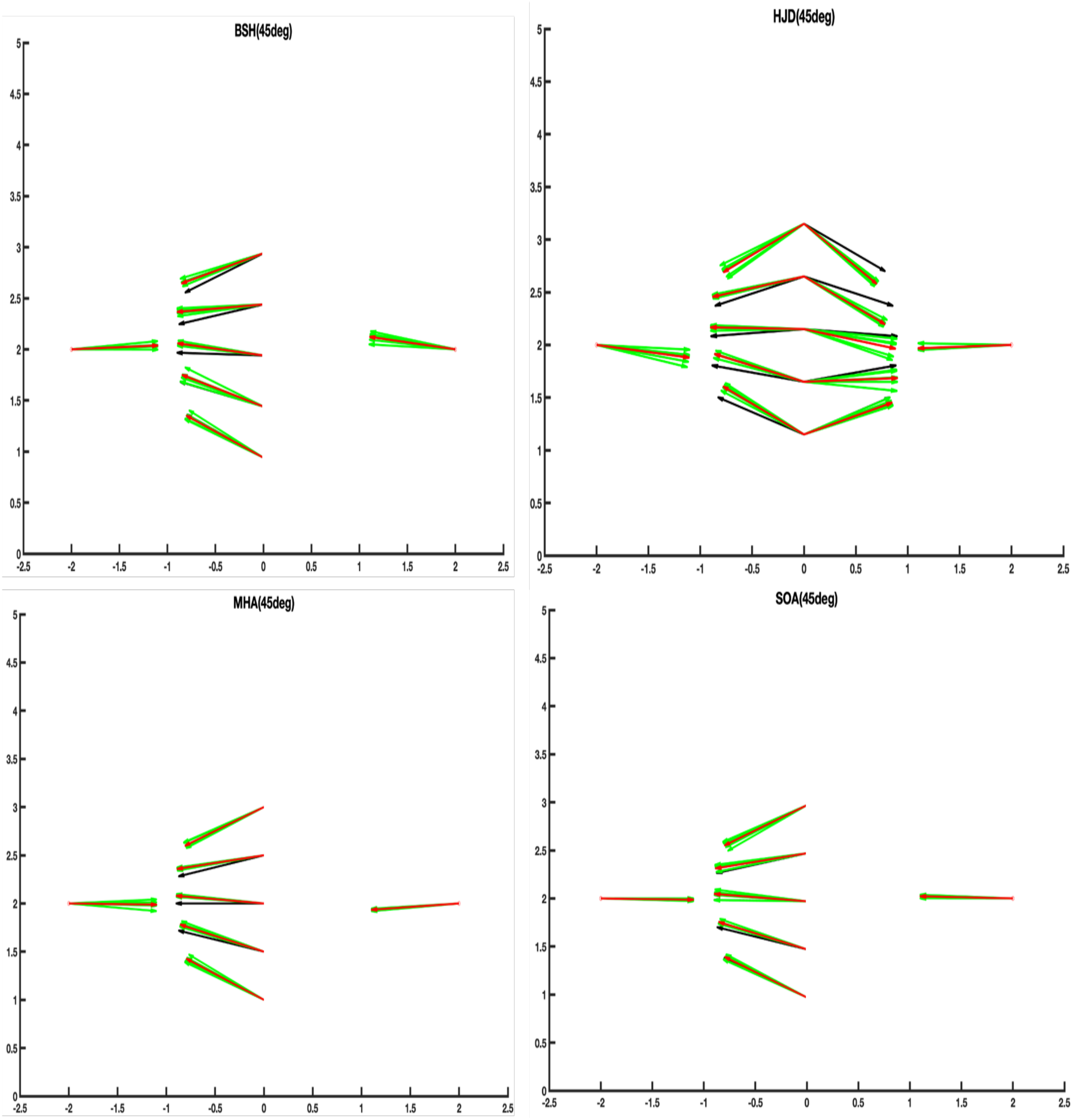


Figure 24 Measurements of the frontoparallel point are obtained as a series of pointings from the sagittal midline to the sides. (45 degree)- Black arrows point to the landmark (real angles), the green arrows show different trials, red ones show the average of measurements from green ones

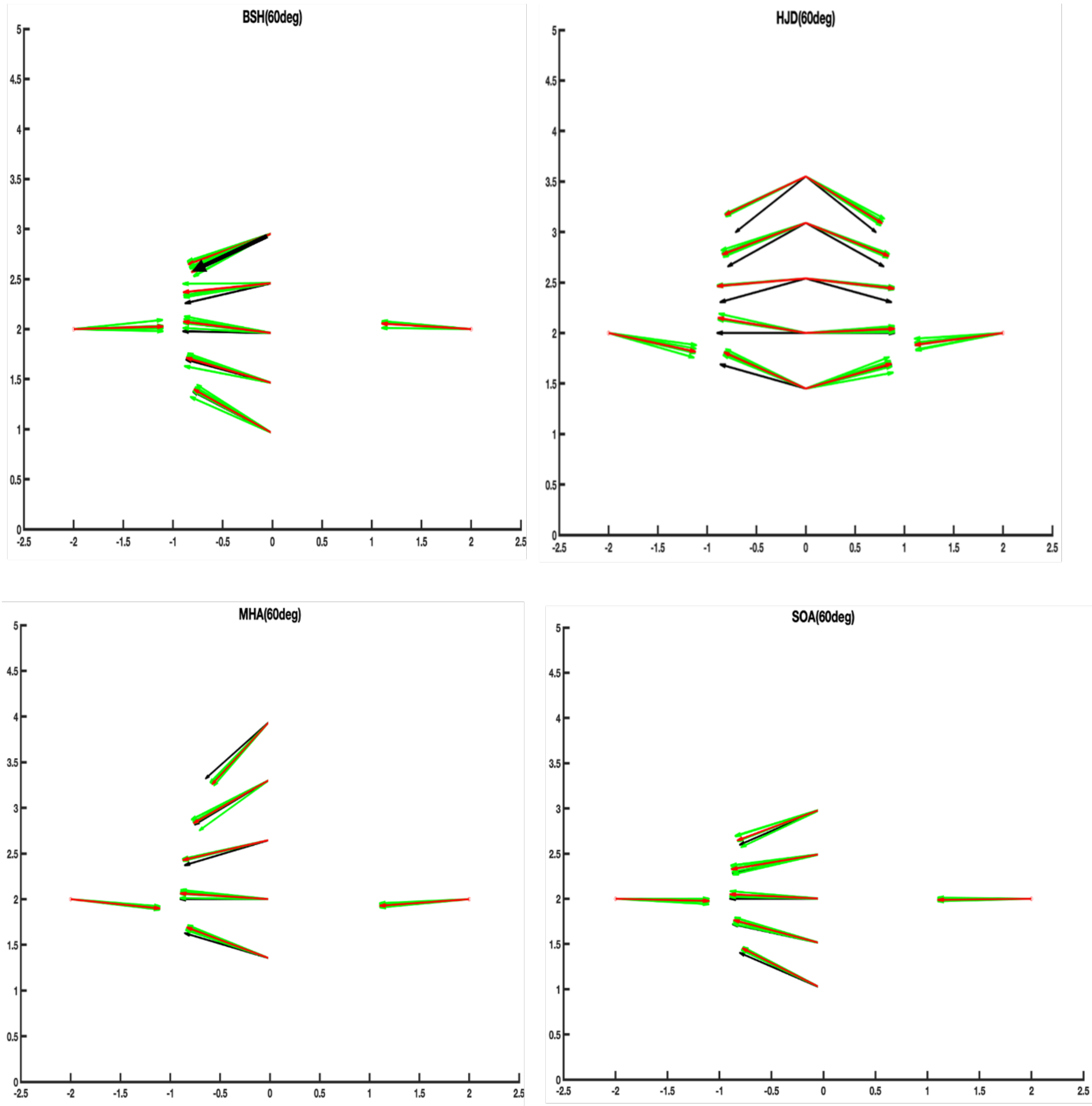


Figure 25 Measurements of the frontoparallel point are obtained as a series of pointings from the sagittal midline to the sides. (60 degree) Black arrows point to the landmark (real angles), the green arrows show different trials, red ones show the average of measurements from green ones

The following figures also show that the average direction pointing as a function of distance. These figures also confirm the finding from the previous experiment that this exocentric pointing task does not reveal any consistent pattern meaning that geodesic is not a circular arc.

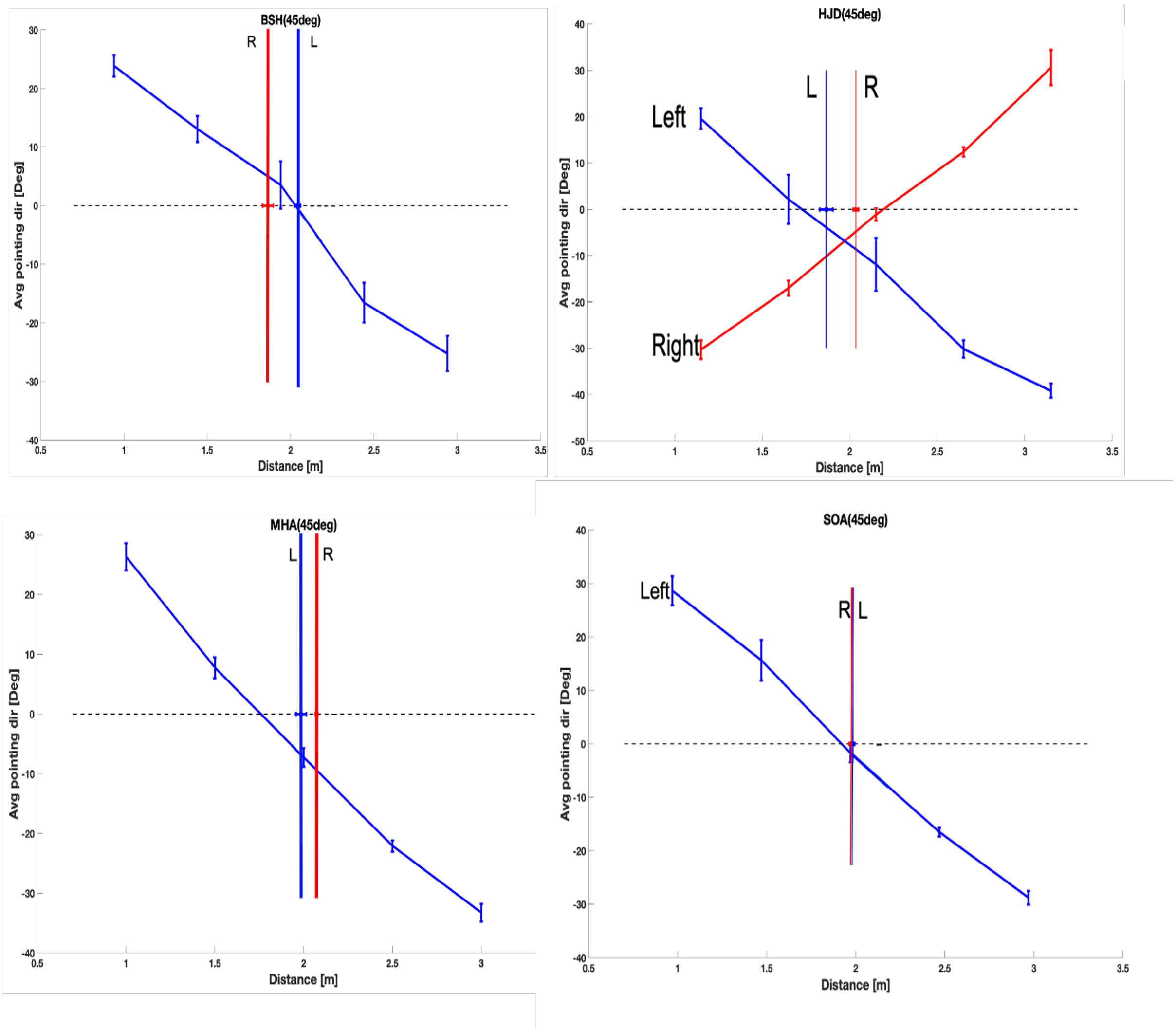


Figure 26 Average pointing direction to distance. Red lines indicate the condition when the targets are at left side of the subjects and blue is for the right condition (45 deg)

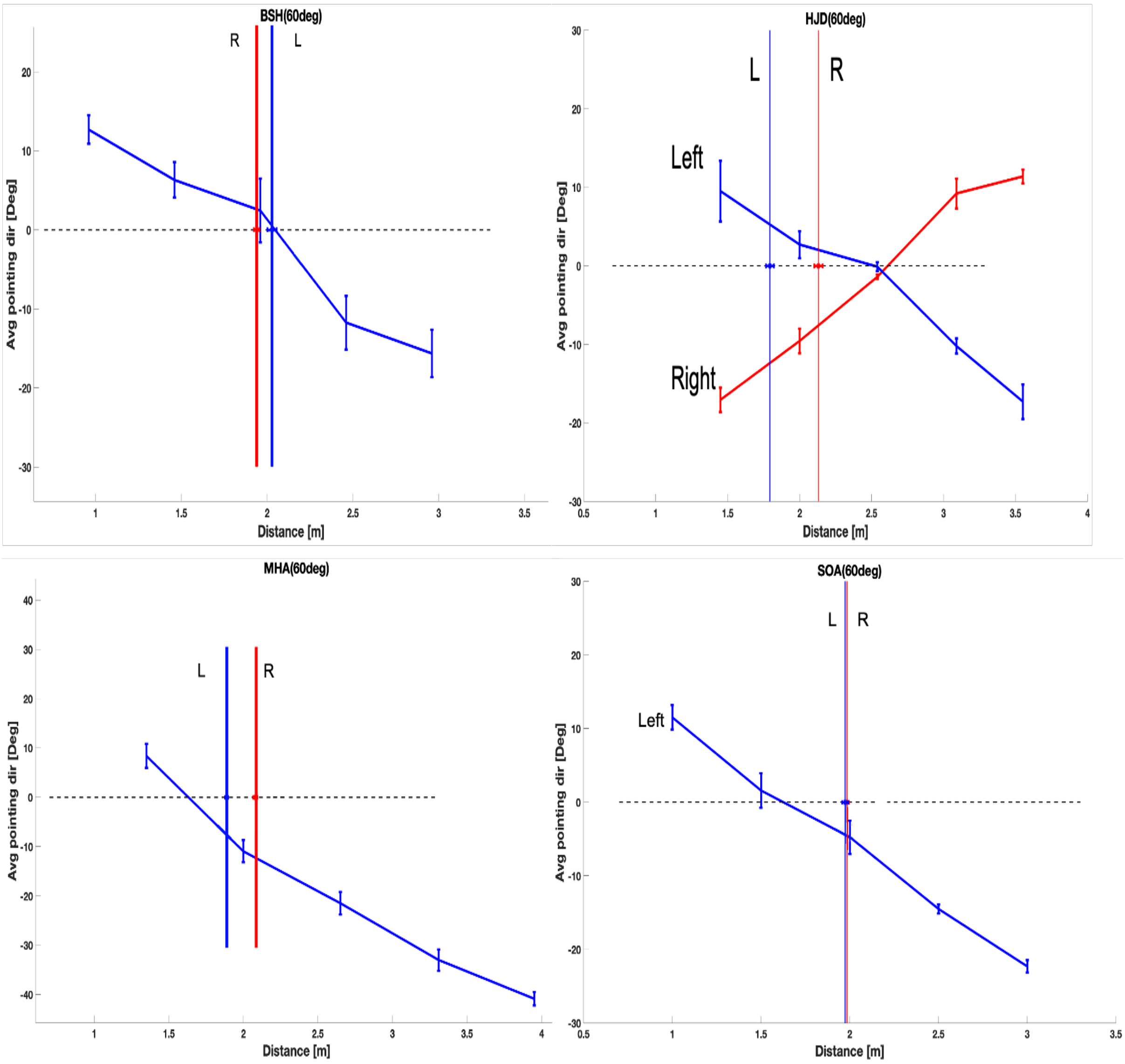


Figure 27Average pointing direction to distance. Red lines indicate the condition when the targets are at left side of the subjects and blue is for the right condition (60 deg)

Since this analysis is performed separately for the rightward and leftward pointing, we omitted the indices r, l. We then considered two points p_i and p_j with angles going into the opposite directions, i.e., $\alpha_i < 0$ and $\alpha_j > 0$. Normally, we will use the closest two points satisfying this condition. From these, we calculate an estimate of Δd by linear interpolation:

$$\Delta d^* = p_i - \alpha_i \frac{p_j - p_i}{\alpha_j - \alpha_i}$$

The confidence interval of the estimate Δd^* can be approximated by

$$\mathsf{sdv}(\Delta d^*) \approx \frac{1}{2}(\mathsf{sdev}(\alpha_i) + \mathsf{sdev}(\alpha_j)) \left| \frac{p_j - p_i}{\alpha_j - \alpha_i} \right| .$$

For each subject and the two values of x (resulting from the 45- and 60-degree conditions), we now have two predictions of the FPP offset called $\Delta d_{l/r}$ and two measured estimates called $\Delta d^*_{l/r}$ together with their standard deviations. The following figures show the plot point-pairs as indicated for all subjects and the two angle conditions. If subjects adjust veridically (FPP offset zero), all points should fall close the horizontal axis. If the theory holds, however, points should fall close to the diagonal.

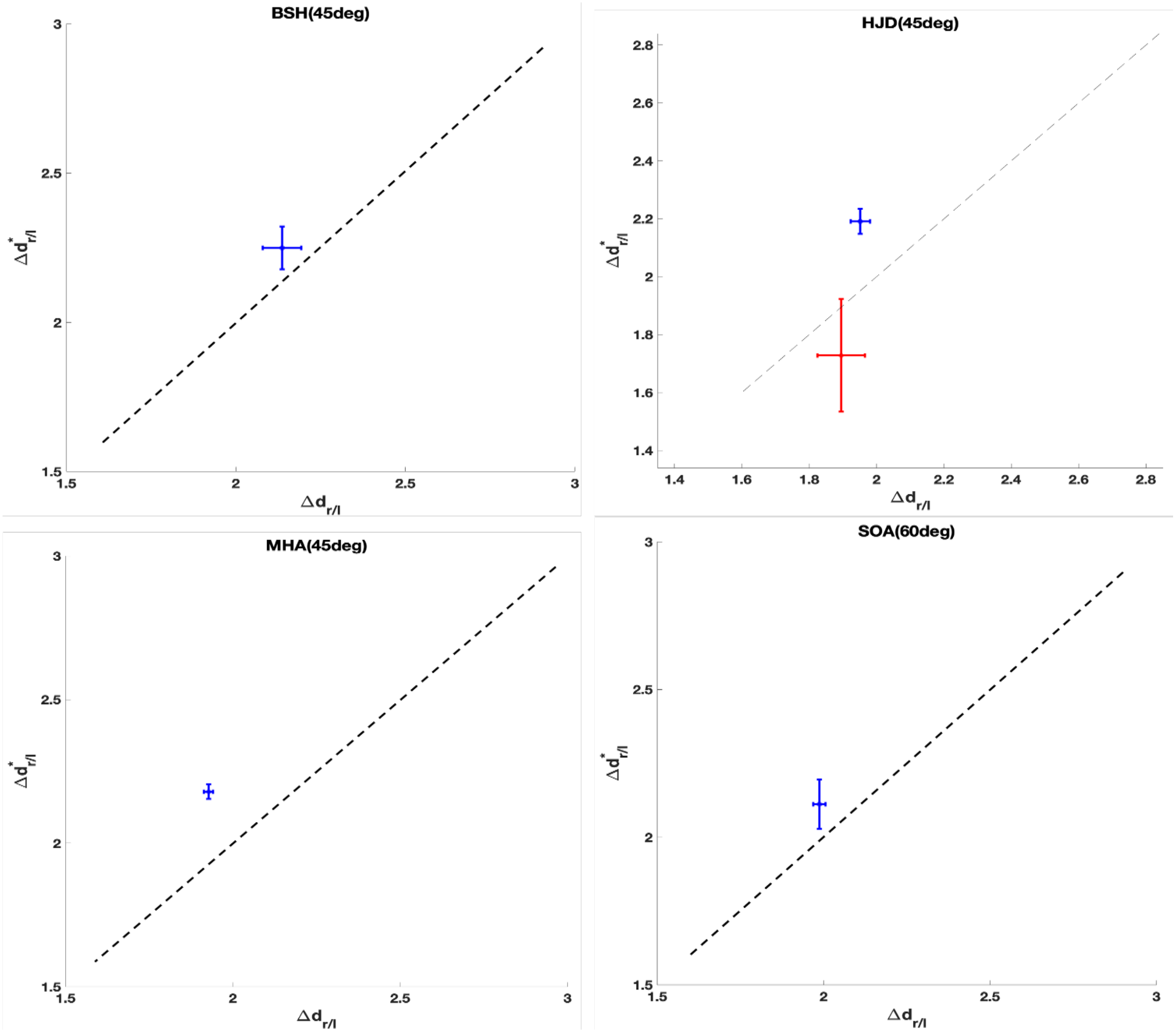


Figure 28 FPP offset to its estimation (45 degree)

+

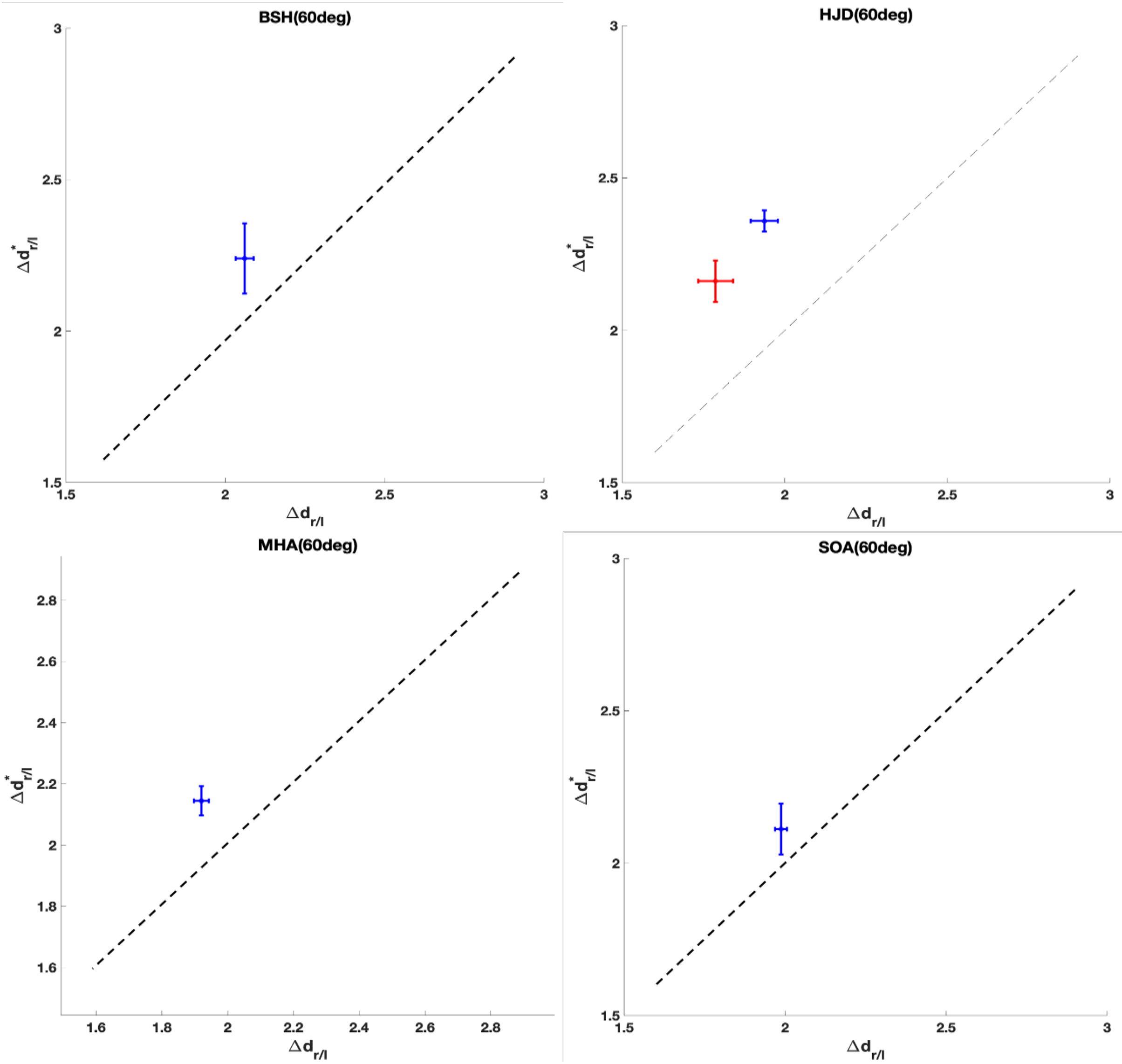


Figure 29 FPP offset to its estimation (60 degree)

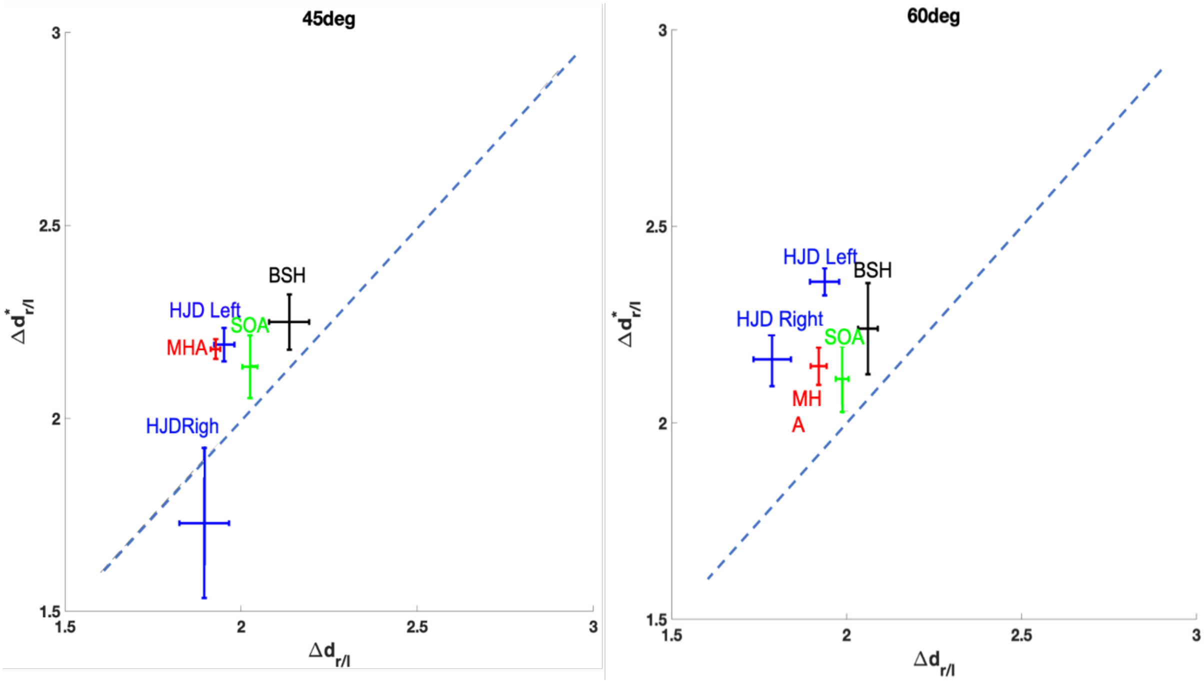


Figure 30 FPP offset to its estimation for all subjects (45 and 60 degree)

The figure above illustrates the same results for all the subjects. The calculation of the correlation between the predicted FPP offset and measured estimates and its comparison with horizontal line confirms that they are significantly different, and they are tended to stay close to the diagonal line meaning the experiment hypothesis is correct.

Result 3:

As it was discussed in detail the goal and instruction of this last experiment, we measured the tangent to the geodesics at upper eye level vertically. As it can be observed in the Figure 31 the deviations of pointed angles from veridical angles called systematic errors have been changed by their signs from positive to negative when subjects point to the upper floors than the one that is close to the eye level namely 4rd floor. This result also illustrates that all four observers show qualitatively (and indeed quantitatively) similar behaviour. This means that all of them have pointed to the targets of the upper floors exceeding the veridical angles drastically (-5 to -20 degrees), while they were close to the veridical for near targets.

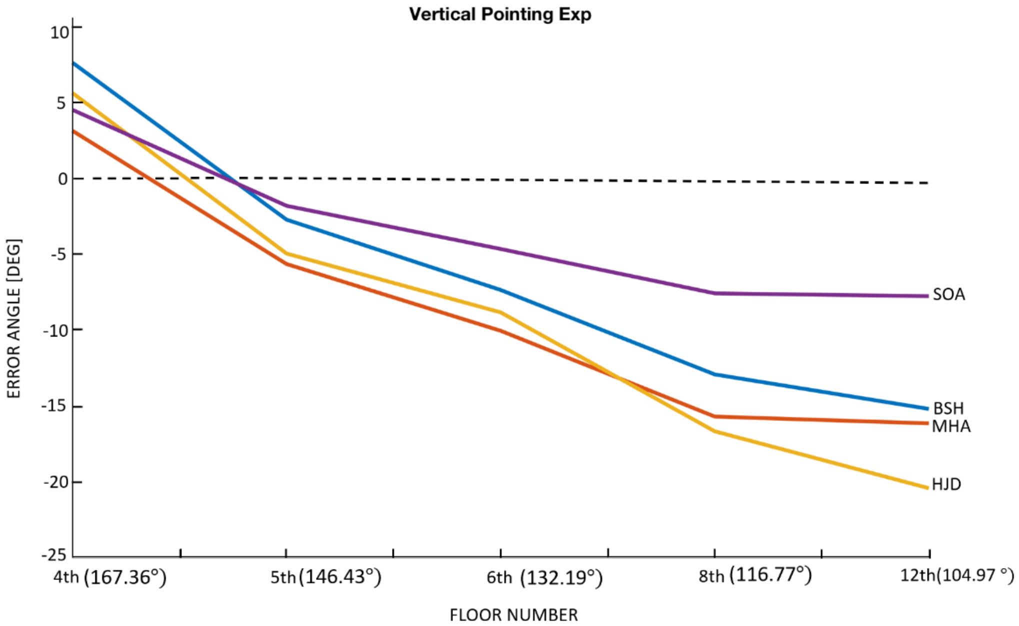


Figure 31 Exp3 results

Chapter V: Discussions

The relationship between physical and visually perceived space is an old philosophical question which has been discussed for centuries. Since the beginning of 20^{th} century, the investigation of the geometric relationship between perceived and physical spaces came to the focus point of many researchers and mathematicians. Using the tools of Euclidean geometry, it was assumed that the physical spatial relations, over the range relevant for human observers, could be perfectly described and a natural assumption could be to also consider the perception of spatial layout as Euclidean. In the Introduction we reported that how the relevant research conducted in the last century have found that the perceived length of a physical interval is influenced by its position and orientation in space, and in contrary to the axioms of Euclidean geometry.

Thanks to the emergence of new definitions of non-Euclidian geometries, they had more descriptive tools to formulate the geometry of visually perceived space and further its relationship with veridical space. Several researches in the beginning of the second half of the last century (i.e., (Blank 1958, 1978), Indow et al. (1962)) came to a common agreement that the geometry of visual space at least of the horizontal plane at eye level is not only non-Euclidian but also is governed by a Riemannian metric with constant negative curvature. Koenderink et al. (2000) doubted on this agreement that the space is of constant curvature and assumed for the first time that the space is Riemannian, and that it is isotropic. Therefore, they conducted an experiment to determine the intrinsic curvature as a function of distance from the observer.

The main motivation of this study was first to repeat and then to improve the exocentric poiting task experiment introduced by Koenderink et al., (2000, 2003). Initially, we had the plan to have additional tangent measurements along a given geodesic to assess its detailed shape. Besides, in a second experiment, we wanted to address the role of additional objects placed in the environment. However, our results did not repeat the findings of these papers and therefore we had to deviate from our predefined line of research and investigate more the possible reasons that could justify these discrepancies.

In the first experiment, we have repeated the exocentric pointing task introduced by Koenderink et al., (2000, 2003) that study exocentric visual directions to targets that are opposite to a pointer relative to the observer. Our experimental setup bears a close resemblance to the one that offered in these papers as explained in the previous chapters. Although we had minor modifications over this experiment paradigm, the purpose of our study was the same as this original papers. However, our results have not confirmed previous research on this exocentric pointing tasks.

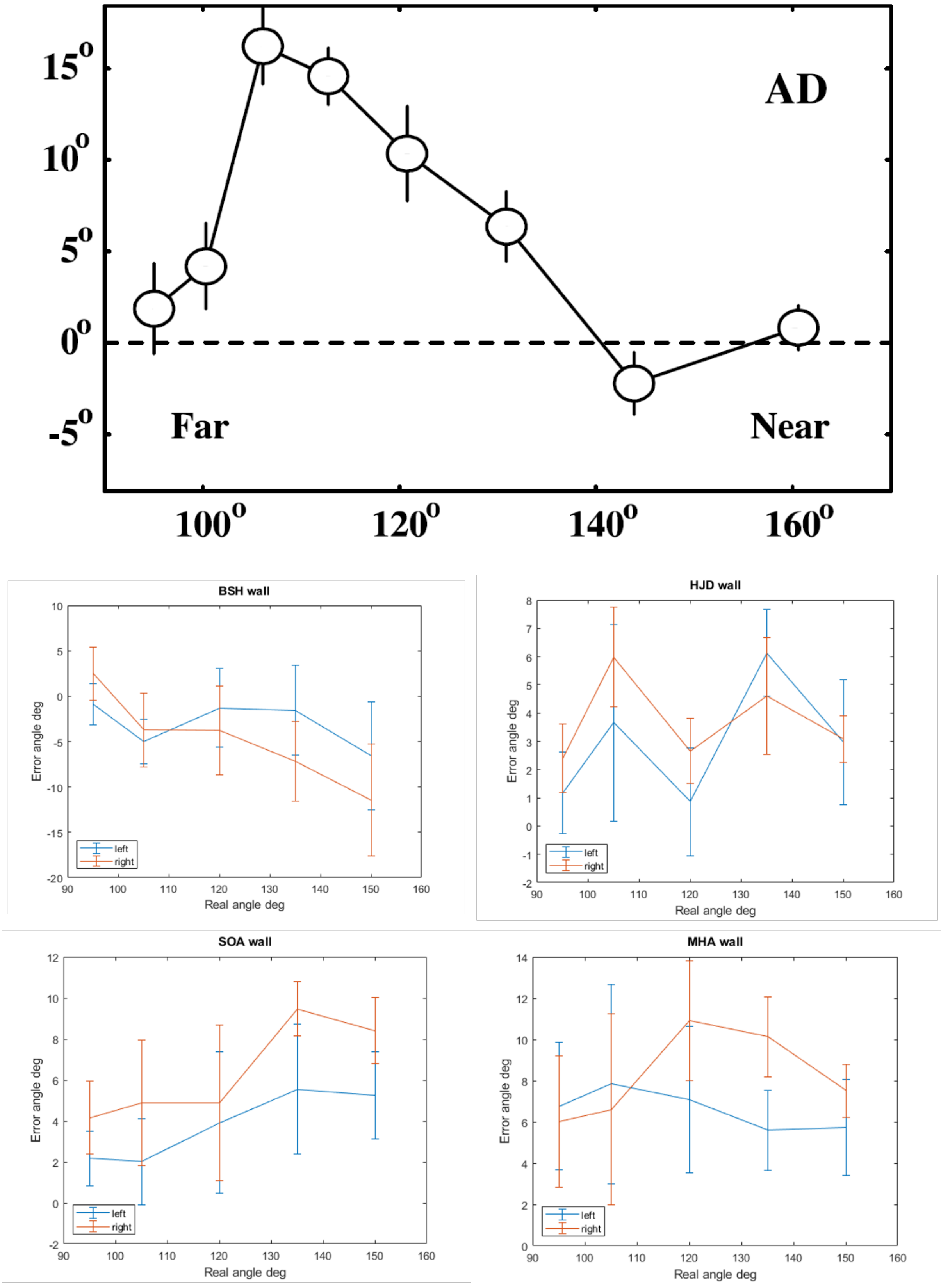


Figure 32 Comparison of the pattern of systematic errors with the original experiment

As it can be seen in the Figure 32 our findings differ qualitatively (and indeed quantitatively) from the results provided in these research papers. While our results show that every subject has its own unique pattern for these symmetric errors, in the experiment conducted by Koenderink (2003) all three observers show similar behaviour. In other words, they observed that for the medium distances (rT) all observers exceed the veridical angle sometimes severely (10–15 degree), for near and far targets they are close to veridical or even show signs of underpointing.

The difference between expected and obtained results may be due to the inherent dissimilarities between the experiment setups. Although we struggled to be loyal to the original experiment setup, we had no other choice but to adapt the experiment based on our own limitations. The most important difference between these two studies was the fact that we performed our study indoor and in one of the university buildings with a large hall. Due to some inherent difficulties such as more frequent cloudy and even rainy days than sunny ones in Tübingen, not having a large open area with no obvious landmarks within the university campus (for instance we ran a set of experiment on the roof of one multistorey car parking but decided to discard and exclude it from the study due to having park marks that were thought to be used as the depth cues for the subjects). The other distinction could be the idea of performing the original experiment in an unkept meadow with weeds growing up to knee and even waist height. Thus, in the experiment done by Koenderink et al., (2003) it was typically not possible to see the base of the tripods meet the ground plane while in our experiment the base of tripods were visible and could be used by subjects as another cue together with the cues that a hall with columns and walls could provide.

Durgin et al., (2009) showed that participants adjust their responses according to the experimental hypothesis that were inferred when the experiment explanation can induce a demand characteristic such that the subjects could guess the experiment's hypothesis. In the experiment done by Koenderink et al., not only the subjects had a chance to guess the experiment's results but also there were the authors of the paper suggesting that they were aware of all the details of the experiment and its expectations. In contrast, we had four subjects two naïve and two non-naïve to the experiment's hypothesis to be able to test if there is any difference between the results of these two groups. Although we did not find any significant difference in the results of these two groups, it can be still argued that in Koenderink's results there is a vivid evidence that responses are biased due to the familiarity of subjects (authors) with the hypothesis of the experiment.

Fristone and Scholl in 2016 argued that giving participants immediate and repeated feedback can influence the participant's response strategy to emphasize accuracy based on real measurements, rather than on subjective feelings of difficulty or another factor that could be guiding responses (Firestone and Scholl, 2016). This might be taken into the consideration in both experiments to have more precise results so that they could show clearer image of visual space perception of subjects. Another cue that could influence our experiment as well as the original one, was the fact that the pointing task was performed in order from the near to far targets. In this case, subjects after several trials might have learned that for pointing to the further targets, they needed to change the pointer but at the same direction. For instance, they might have realized that pointing from 2m target to 20m target means to slightly change the pointer to the right each time. This could be avoided by randomizing the order of the pointing

task; however, this could have consumed more time than it did. (Experiment one took three hours in average for each subject that means adding the setup time that could exceed the amount of time that we had permission to perform the experiment in the mentioned building at each day.)
In the second experiment as it was discussed we tested an implicit prediction from the Koendering et al. (2000) paper in which they had suggested that the geodesics from pointer to the target could be modelled as circular arcs to which the pointer is the tangent. This itself suggests that an intermediate point on an assumed geodesic has a pointing direction that is frontoparallel to the observer. Then we observed that the geodesic was not a circular arc and consequently we excluded the prediction that the frontoparallel point *I* is indeed in the midsagittal plane. Therefore, we calculated two predictions for the deviation of *I* from the directed connections. The with series of pointing from the sagittal midline to the side we measured the frontoparallel points.
Having calculated the predictions and measured the frontoparallel points, for each subject we had a point-pairs as depicted in the following figure. By testing the correlation between predicted and measured values that were confirmed falling close to the diagonal line ($\Delta d^*_{l/r} = \Delta d_{l/r}$) the expected effect was demonstrated.

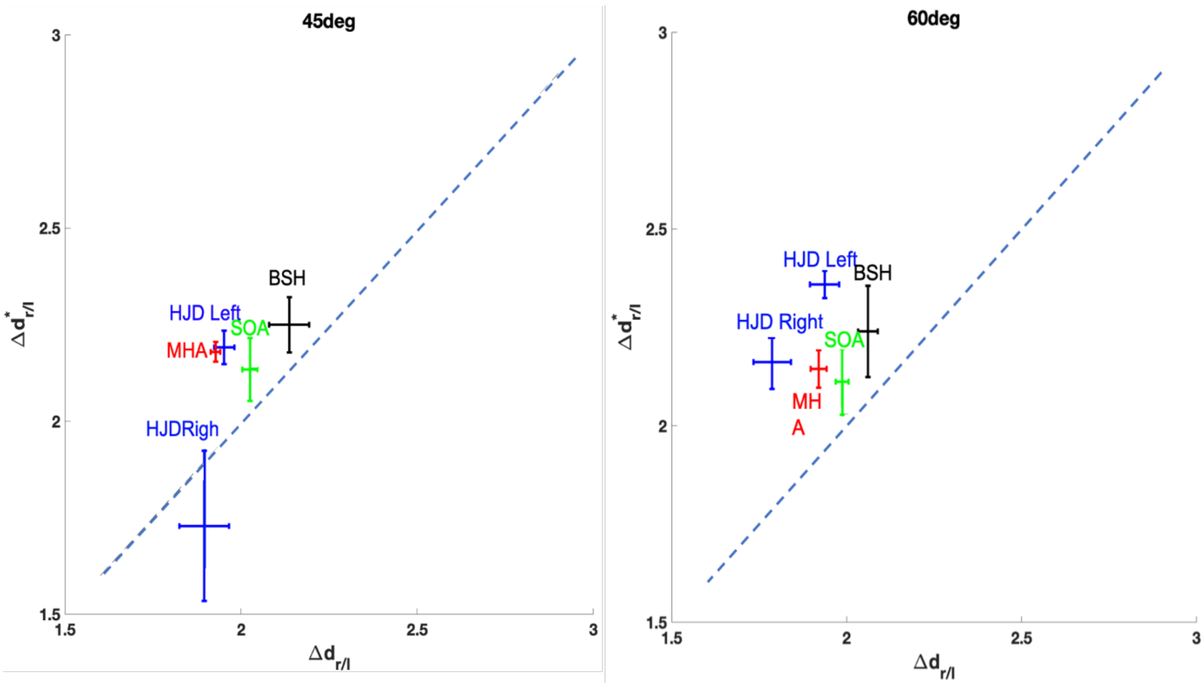


Figure 33 FPP offset to its estimation for all subjects (45 and 60 degree)

In the last experiment that we measured the tangent to the geodesics at upper eye level vertically the results demonstrate that all four observers show qualitatively (and indeed quantitatively) similar behaviour. This means that all of them have pointed to the targets of the upper floors exceeding the veridical angles drastically (-5 to -20 degrees), while they were close to the veridical for near targets.

Apart from mentioned possible biases or cues that might have affected the results some of the inherent experiment setup could have influence on the results. The most obvious one would be to the fact that we did not have any chance to highlight our landmarks on the targets as we did in previous experiment. As it was discussed in the previous chapter participants were told to point to a hypothetical line in the middle of balcony of each floor. This could be a problem for lower floors, but it was indeed a source of confusion for the participants when they were pointing to the upper floors. They had to both find the right order of floor number and at the same time calculate roughly the midline of its balcony. The second intrinsic source of bias in our experiment setup was obviously the weather condition. Since the experiment was conducted outside the light could not be fixed. Although we excluded the rainy days for our experiment, we still did not have control over the sun light in partly cloudy days and consequently during the experiment even for one subject this factor could vary.

In conclusion and being aware of all sources of biases we still can show that our result in this experiment shows the deviation from veridical angles in this exocentric pointing task demonstrate that the visually perceived space is different than physical space. However, in contrast to the mentioned papers (Koenderink et al., 2000; 2003) we did not repeat their findings that suggest that visual space is governed by a Riemannian metric with varying intrinsic curvature.

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