Gödel's First Incompleteness Theorem

Theorem (First Incompleteness Theorem; Gödel 1931)

Assume, PA is consistent. Then, there is a sentence φ such that:

1. $\mathsf{PA} \nvdash \varphi$;
2. If $\mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner\varphi\urcorner) \Rightarrow \mathsf{PA} \vdash \varphi$, then $\mathsf{PA} \nvdash \neg\varphi$.

Proof.

According to the diagonalization lemma, there is a sentence φ such that

$$\mathsf{PA} \vdash \varphi \leftrightarrow \neg\mathrm{B}_{\mathsf{PA}}(\ulcorner\varphi\urcorner). \qquad (*)$$

1. Assume $\mathsf{PA} \vdash \varphi$. With (1) we have $\mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner\varphi\urcorner)$. With $(*)$ it follows $\mathsf{PA} \vdash \neg\mathrm{B}_{\mathsf{PA}}(\ulcorner\varphi\urcorner)$ in contradiction to the consistency of PA.
2. Assume $\mathsf{PA} \vdash \neg\varphi$. With $(*)$ we have $\mathsf{PA} \vdash \neg\neg\mathrm{B}_{\mathsf{PA}}(\ulcorner\varphi\urcorner)$ and also $\mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner\varphi\urcorner)$. Because of the additional premise this gives $\mathsf{PA} \vdash \varphi$, again in contradiction to the consistency of PA.

First Incompleteness Theorem: generic form

- The premise $\mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner\varphi\urcorner) \Rightarrow \mathsf{PA} \vdash \varphi$ in the second case corresponds to the ω-*consistency* which was assumed by Gödel in his original paper.
- In 1936, B. J. Rosser found a trick to avoid this condition, using a modified proof predicate Bew^R "on top" of Gödel's proof.
- The result can be extended to any consistent, *recursive* extension of *PA*:

Theorem (First Incompleteness Theorem)

Assume, that T is a consistent, recursive extension of PA. Then, there is a sentence φ such that:

1. $T \nvdash \varphi$;
2. $T \nvdash \neg\varphi$.

Gödel's second incompleteness theorem

- Gödel's second incompleteness theorem says that a theory, which has at least the expressive power of Peano Arithmetic, cannot prove its own consistency.
- Using the techniques developed so far, consistency of a theory T can be easily expressed as:

$$\mathrm{Con}_T \iff \neg \mathrm{B}_T(\ulcorner \Lambda \urcorner)$$

where Λ is an arbitrary contradictory (false) formula, for instance, $0 = s(0)$.

- We say that a theory does not prove it own consistency if we have:

$$T \nvdash \mathrm{Con}_T.$$

The idea of the proof of Gödel II

- First we consider, again, only PA.
- In a sloppy formulation, the idea for the proof of the second incompleteness theorem is to formalize the proof of the first incompleteness theorem in PA.

1. If $\mathsf{PA} \nvdash \varphi$, PA is obviously consistent (as an inconsistent theory proves every formula). Thus:

$$\mathsf{PA} \nvdash \varphi \Longrightarrow \mathsf{PA} \text{ is consistent.}$$

2. The first incompleteness theorem states, for the chosen φ:

$$\mathsf{PA} \text{ is consistent} \Longrightarrow \mathsf{PA} \nvdash \varphi.$$

- The formalization of both arguments *within* PA will show that this φ is equivalent to the consistency statement of PA:

$$\mathsf{PA} \vdash \neg \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \leftrightarrow \mathrm{Con}_{\mathsf{PA}}$$

$$\mathsf{PA} \vdash \varphi \leftrightarrow \mathrm{Con}_{\mathsf{PA}}.$$

Provability conditions

- For the proof of the first incompleteness theorem we used the following property of B:

$$\mathsf{PA} \vdash \varphi \implies \mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner\varphi\urcorner) \quad (1)$$

- For the proof of the second incompleteness theorem, we need the two additional properties of B_{PA}:

$$\mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner\varphi\urcorner) \to \mathrm{B}_{\mathsf{PA}}(\ulcorner\mathrm{B}_{\mathsf{PA}}(\ulcorner\varphi\urcorner)\urcorner) \quad (2)$$

$$\mathsf{PA} \vdash [\mathrm{B}_{\mathsf{PA}}(\ulcorner\varphi\urcorner) \wedge \mathrm{B}_{\mathsf{PA}}(\ulcorner\varphi \to \psi\urcorner)] \to \mathrm{B}_{\mathsf{PA}}(\ulcorner\psi\urcorner) \quad (3)$$

- (2) and (3) do not follow any longer directly from the representability theorem. But they can be proven for B_{PA} (with some hard work).
- The three conditions are called *Hilbert-Bernays-Löb derivablity conditions*. They can be studied independently, and in an abstract form they are the base of *provability logic*.

Gödel's second incompleteness theorem

Theorem (Second incompleteness theorem)

Assume PA is consistent. Then we have:

$$\mathsf{PA} \nvdash \mathrm{Con}_{\mathsf{PA}}.$$

Proof of Gödel's second incompleteness theorem

- Let φ be such that: $\mathsf{PA} \vdash \varphi \leftrightarrow \neg \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner)$ $(\star)$
$\mathsf{PA} \vdash \Lambda \to \varphi$	Ex-falso-quodlibet
$\mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner \Lambda \urcorner) \to \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner)$	(1) and (3)
$\mathsf{PA} \vdash \neg \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \neg \mathrm{B}_{\mathsf{PA}}(\ulcorner \Lambda \urcorner)$	Contrapositive
$\mathsf{PA} \vdash \varphi \to \neg \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner)$	$(\star)$
$\mathsf{PA} \vdash \varphi \to \neg \mathrm{B}_{\mathsf{PA}}(\ulcorner \Lambda \urcorner)$	Logical reasoning
$\mathsf{PA} \vdash \varphi \to \mathrm{Con}_{\mathsf{PA}}$	Definition of $\mathrm{Con}_{\mathsf{PA}}$
$\mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \neg \varphi$	Contrapositive of $(\star)$
$\mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \urcorner) \to \mathrm{B}_{\mathsf{PA}}(\ulcorner \neg \varphi \urcorner)$	(1) and (3)
$\mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \mathrm{B}_{\mathsf{PA}}(\ulcorner \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \urcorner)$	(2)
$\mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \mathrm{B}_{\mathsf{PA}}(\ulcorner \neg \varphi \urcorner)$	Logical reasoning
$\mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \wedge \neg \varphi \urcorner)$	(1), (3) and logical reasoning
$\mathsf{PA} \vdash \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \mathrm{B}_{\mathsf{PA}}(\ulcorner \Lambda \urcorner)$	Definition of Λ
$\mathsf{PA} \vdash \neg \mathrm{B}_{\mathsf{PA}}(\ulcorner \Lambda \urcorner) \to \neg \mathrm{B}_{\mathsf{PA}}(\ulcorner \varphi \urcorner)$	Contrapositive
$\mathsf{PA} \vdash \mathrm{Con}_{\mathsf{PA}} \to \varphi$	Definition of $\mathrm{Con}_{\mathsf{PA}}$ and $(\star)$
- As $\mathsf{PA} \nvdash \varphi$ we have also $\mathsf{PA} \nvdash \mathrm{Con}_{\mathsf{PA}}$.

Gödel's second incompleteness theorem; generic version

Theorem (Second incompleteness theorem; Gödel 1931)

Assume, that T is a consistent, *recursive* extension of PA. Then

$$T \nvdash \mathrm{Con}_T.$$

Why reasoning in PA about PA?

- Assume, the second incompleteness theorem would not hold, and it would be the case that PA $\vdash \mathrm{Con}_{\mathsf{PA}}$.
- Obviously, such a proof would not give any evidence for the consistency of PA: if PA would be incosistent, every formula would be provable, in particular also $\mathrm{Con}_{\mathsf{PA}}$.
- The significance of the second incompleteness theorem (as given here) is based on an immediate corollary: if PA cannot prove its consistency, no *weaker* theory—in particular, any subsystem of PA—could do so.
- But this was the idea in **Hilbert's programme**: using *finitistic mathematics*—which is is supposed to be a subsystem of PA—to prove the consistency of PA (and other theories).

Consistency Proofs after Gödel

- For PA, we may consider the following three alternative approaches (all of them already discussed by Gödel as early as 1938):
  1. Intuitionistic Arithmetic: double negation interpretation. (Kolmogorov 1925; Gödel 1933; Gentzen 1936)
  2. Primitive-recursive arithmetic with *transfinite* induction up to the ordinal ε_0 (Gentzen 1936)
  3. Functionals of higher type: *Gödel's $\mathcal{T}$; Dialectica interpretation* (Gödel 1958)
- What about stronger systems, first of all *Analysis*?
- In the following we will pursue a little bit further *Ordinal Analysis* in Gentzen-style proof theory.

  The following slides are taken with permission from a course given by Michael Rathjen in 2005.

Sequent Calculus

A **se e t** is an expression $\Rightarrow$ where and are finite sequences of formulae $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$, respectively.
$\Rightarrow$ is read, informally, as yields or, rather, the conjunction of the A_i yields the disjunction of the B_j.

In particular,

- If is empty, the sequent asserts the disjunction of the B_j.
- If is empty, it asserts the negation of the conjunction of the A_i.
- if and are both empty, it asserts the impossible, i.e. a contradiction.

We use upper case Greek letters $, \; , \Lambda, \; , \; \ldots$ to range over finite sequences of formulae.

Sequent Calculus

Identity Axiom

$$A \Rightarrow A$$

where A is any formula. In point of fact, one could limit this axiom to the case of atomic formulae A.

CUT

$$\frac{\Rightarrow \; , A \qquad A, \Lambda \Rightarrow}{, \Lambda \Rightarrow \; ,} \text{ Cut}$$

A is called the cut formula of the inference.

Sequent Calculus

Structural Rules — Exchange, Weakening, Contraction

$$\frac{, A, B, \Lambda \Rightarrow}{, B, A, \Lambda \Rightarrow}\ {}_{l} \qquad \frac{\Rightarrow \quad , A, B, \Lambda}{\Rightarrow \quad , B, A, \Lambda}\ {}_{r}$$

$$\frac{\Rightarrow}{, A \Rightarrow}\ {}_{l} \qquad \frac{\Rightarrow}{\Rightarrow \quad , A}\ {}_{r}$$

$$\frac{, A, A \Rightarrow}{, A \Rightarrow}\ \mathcal{C}_l \qquad \frac{\Rightarrow \quad , A, A}{\Rightarrow \quad , A}\ \mathcal{C}_r$$

Sequent Calculus

Negation

$$\frac{\Rightarrow \quad , A}{\neg A, \quad \Rightarrow}\ \neg \mathrm{L} \qquad \frac{B, \quad \Rightarrow}{\Rightarrow \quad , \neg B}\ \neg \mathrm{R}$$

Implication

$$\frac{\Rightarrow \quad , A \qquad B, \Lambda \Rightarrow}{A \to B, \quad , \Lambda \Rightarrow \quad ,}\ \to \mathrm{L} \qquad \frac{A, \quad \Rightarrow \quad , B}{\Rightarrow \quad , A \to B}\ \to \mathrm{R}$$

Sequent Calculus

Conjunction

$$\frac{A, \quad \Rightarrow}{A \wedge B, \quad \Rightarrow} \wedge L1 \qquad \frac{B, \quad \Rightarrow}{A \wedge B, \quad \Rightarrow} \wedge L2$$

$$\frac{\Rightarrow \quad , A \qquad \Rightarrow \quad , B}{\Rightarrow \quad , A \wedge B} \wedge R$$

Disjunction

$$\frac{A, \quad \Rightarrow \qquad B, \quad \Rightarrow}{A \vee B, \quad \Rightarrow} \vee L$$

$$\frac{\Rightarrow \quad , A}{\Rightarrow \quad , A \vee B} \vee R1 \qquad \frac{\Rightarrow \quad , B}{\Rightarrow \quad , A \vee B} \vee R2$$

Sequent Calculus

Quantifiers

$$\frac{(t), \quad \Rightarrow}{\forall x \quad (x), \quad \Rightarrow} \forall L \qquad \frac{\Rightarrow \quad , \quad (a)}{\Rightarrow \quad , \forall x \quad (x)} \forall R$$

$$\frac{(a), \quad \Rightarrow}{\exists x \quad (x), \quad \Rightarrow} \exists L \qquad \frac{\Rightarrow \quad , \quad (t)}{\Rightarrow \quad , \exists x \quad (x)} \exists R$$

In $\forall$L and $\exists$R, t is an arbitrary term. The variable a in $\forall$R and $\exists$L is an eigenvariable of the respective inference, i.e. a is not to occur in the lower sequent.

Sequent Calculus

The formulae in a logical inference marked blue are called the minor formulae of that inference, while the red formula is the principal formula of that inference. The other formulae of an inference are called side formulae.

A proof (aka deduction or derivation) $\mathcal{D}$ is a tree of sequents satisfying the following conditions:

- The topmost sequents of $\mathcal{D}$ are identity axioms.
- Every sequent in $\mathcal{D}$ except the lowest one is an upper sequent of an inference whose lower sequent is also in $\mathcal{D}$.

Sequent Calculus

e ase

The intuitionistic sequent calculus is obtained by requiring that all sequents be intuitionistic. A sequent $\Rightarrow$ is said to be intuitionistic if consists of at most one formula.

Specifically, in the intuitionistic sequent calculus there are no inferences corresponding to contraction right or exchange right.

Sequent Calculus

Our first example is a deduction of the law of excluded middle.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A, \neg A}\,\neg\mathrm{R}}{\Rightarrow A,\ A \vee \neg A}\,\vee\mathrm{R}}{\Rightarrow A \vee \neg A,\ A}\,{}^{r}}{\Rightarrow A \vee \neg A,\ A \vee \neg A}\,\vee\mathrm{R}}{\Rightarrow A \vee \neg A}\,\mathcal{C}_r
$$

Notice that the above proof is not intuitionistic since it involves sequents that are not intuitionistic.

Sequent Calculus

The second example is an intuitionistic deduction.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(a) \Rightarrow \ (a)}{(a) \Rightarrow \exists x \ (x)}\,\exists\mathrm{R}}{\neg\exists x \ (x),\ (a) \Rightarrow}\,\neg\mathrm{L}}{(a),\ \neg\exists x \ (x) \Rightarrow}\,{}^{l}}{\neg\exists x \ (x) \Rightarrow \neg \ (a)}\,\neg\mathrm{L}}{\neg\exists x \ (x) \Rightarrow \forall x \neg \ (x)}\,\forall\mathrm{R}}{\Rightarrow \neg\exists x \ (x) \to \forall x \neg \ (x)}\,{\to}\mathrm{R}
$$

Cut Elimination

t limi atio (Gentzen's Hauptsatz)

If a sequent $\Rightarrow$ is provable,
then it is provable without cuts.

Here is an example of how to eliminate cuts of a special form:

$$\dfrac{\dfrac{A, \ \Rightarrow \ , B}{\Rightarrow \ , A \to B}\to\text{R} \qquad \dfrac{\Lambda \Rightarrow \ , A \qquad B, \ \Rightarrow \Phi}{A \to B, \Lambda, \ \Rightarrow \ , \Phi}\to\text{L}}{, \Lambda, \ \Rightarrow \ , \ , \Phi}\text{Cut}$$

is replaced by

$$\dfrac{\dfrac{\Lambda \Rightarrow \ , A \qquad A, \ \Rightarrow \ , B}{\Lambda, \ \Rightarrow \ , \ , B}\text{Cut} \qquad B, \ \Rightarrow \Phi}{, \Lambda, \ \Rightarrow \ , \ , \Phi}\text{Cut}$$

Cut Elimination

Remarks

- The proof of the cut elimination theorem is rather intricate as the process of removing cuts interferes with contraction.

  The possibility of contraction accounts for the high cost of eliminating cuts. Let $|\mathcal{D}|$ be the height of the deduction $\mathcal{D}$. Also, let $rank(\mathcal{D})$ be supremum of the lengths of cut formulae occurring in $\mathcal{D}$. Turning $\mathcal{D}$ into a cut-free deduction of the same end sequent results, in the worst case, in a deduction of height

$$(rank(\mathcal{D}), |\mathcal{D}|)$$

  where

$$(0, n) = n \qquad (k+1, n) = 4^{\mathcal{H}(\ ,n)}.$$

Cut Elimination

- Cut-free proofs aren't suitable for the mathematical practice. The cut formulae in a proof usually carry the idea of the proof (lemmata). Removing cuts not only makes proofs longer but also renders them less understandable.

Cut Elimination

The Hauptsatz has an important corollary.

The **b orm la ropert**

f a sequent ⇒ is provable then it has a deduction all of whose formulae are subformulae of the formulae of and .

orollar

A contradiction i.e. the empty sequent is not deducible.

Mathematical Theories

While mathematics is based on logic, it cannot be developed solely on the basis of pure logic. What is needed in addition are axioms that assert the existence of mathematical objects and their properties. Logic plus axioms gives rise to (formal) theories such as Peano arithmetic or the axioms of Zermelo-Fraenkel set theory.

Mathematical Theories

What happens when we try to apply the procedure of cut elimination to theories? Well, axioms are poisonous to this procedure. It breaks down because the symmetry of the sequent calculus is lost. In general, we cannot remove cuts from deductions in a theory T when the cut formula is an axiom of T. However, sometimes the axioms of a theory are of bounded syntactic complexity. Then the procedure applies partially in that one can remove all cuts that exceed the complexity of the axioms of T.

Mathematical Theories

This gives rise to

partial t elimi atio .

This is a very important tool in proof theory. For example, it works very well if the axioms of a theory can be presented as atomic intuitionistic sequents (also called Horn clauses), yielding the completeness of Robinsons **resol tio met o .**

Mathematical Theories

Partial cut elimination also pays off in the case of fragments of and set theory with restricted induction schemes, be it induction on natural numbers or sets. This method can be used to extract bounds from proofs of Π^0_2 statements in such fragments.