Advanced Mathematical Methods

WS 2020/21

1 Linear Algebra

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Outline: Linear Algebra

Readings

- Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne Strøm. *Further Mathematics for Economic Analysis.* Prentice Hall, 2008 Chapter 1

Online Resources

MIT course on Linear Algebra (by Gilbert Strang)

- Lecture 21: Eigenvalues and Eigenvectors
  https://www.youtube.com/watch?v=lXNXrLcoerU
- Lecture 22: Powers of a square matrix and Diagonalization
  https://www.youtube.com/watch?v=13r9QY6cmjc
- Lecture 26: Symmetric matrices and positive definiteness
  https://www.youtube.com/watch?v=umt6BB1nJ4w
- Lecture 27: Positive definite matrices and minima – Quadratic forms
  https://www.youtube.com/watch?v=vF7eyJ2g3kU

1.8 Eigenvalues and eigenvectors

assume a scalar λ exists such that

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

λ: eigenvalue
$\boldsymbol{x}$: eigenvector

find λ via the homogenous linear equation system

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

1.8 Eigenvalues and eigenvectors

The properties of a quadratic homogenous linear equation system imply that:

- in any case a solution does exist;
- if $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) \neq 0$, then $\bar{\boldsymbol{x}} = \boldsymbol{0}$ is the trivial solution;
- only if $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$ there is a non-trivial solution.

1.8 Eigenvalues and eigenvectors

Determination of the eigenvalues via *characteristic equation*:

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| \; = \; 0 \quad \Longleftrightarrow \quad (-1)^n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \ldots + \alpha_1 \lambda + \alpha_0 \; = \; 0$$

for every (real or complex) eigenvalue λ_i of the $(n \times n)$-Matrix $\boldsymbol{A}$ we can calculate the respective eigenvector $\boldsymbol{x}_i \neq \boldsymbol{0}$ solving the homogenous linear equation system

$$(\boldsymbol{A} - \lambda_i \boldsymbol{I}) \boldsymbol{x}_i = \boldsymbol{0} \,. \tag{1}$$

The properties of homogenous linear equation systems imply that the solution of eq. (1) is not unambiguous, i.e. for the eigenvalue λ_i we can find infinitely many eigenvectors $\boldsymbol{x}_i$

1.8 Eigenvalues and eigenvectors

$\boldsymbol{A}$ und $\boldsymbol{B}$ (quadratic matrices of order n) are similar if a regular $(n \times n)$ - matrix $\boldsymbol{C}$ exists, such that

$$\boldsymbol{B} = \boldsymbol{C}^{-1} \boldsymbol{A}\, \boldsymbol{C} \ .$$

Special case: **symmetric matrices**
For a symmetric $(n \times n)$-matrix $\boldsymbol{A}$ it holds that the normalized eigenvectors $\tilde{\boldsymbol{x}}_j$ with $j = 1, \ldots, n$ have the property

(1) $\tilde{\boldsymbol{x}}_j' \tilde{\boldsymbol{x}}_j = 1$ for all j and

(2) $\tilde{\boldsymbol{x}}_i' \tilde{\boldsymbol{x}}_j = 0$ for all $i \neq j$.

1.8 Eigenvalues and eigenvectors

Principle axis theorem

collecting the normalized eigenvectors $\tilde{\boldsymbol{x}}_j$ $(j = 1, \ldots, n)$ in a new matrix $\boldsymbol{T} = [\tilde{\boldsymbol{x}}_1 \cdots \tilde{\boldsymbol{x}}_n]$ with the property $\boldsymbol{T}^{-1} = \boldsymbol{T}'$ yields the diagonalization of $\boldsymbol{A}$ as follows:

$$\boldsymbol{D} = \boldsymbol{T}'\boldsymbol{A}\boldsymbol{T} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix}$$

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

1) The product of the eigenvalues of a $(n \times n)$-matrix yields its determinant: $|\boldsymbol{A}| = \prod_{i=1}^{n} \lambda_i$.
2) From 1.) it follows that a singular matrix must have at least one eigenvalue $\lambda_i = 0$.
3) The matrices $\boldsymbol{A}$ and $\boldsymbol{A}'$ have the same eigenvalues.
4) For a non-singular matrix $\boldsymbol{A}$ with eigenvalues λ we have: $|\boldsymbol{A}^{-1} - \frac{1}{\lambda}\boldsymbol{I}| = 0$.
5) Symmetric matrices have only real eigenvalues.
6) The rank of a symmetric matrix $\boldsymbol{A}$ is equal to the number of eigenvalues different from zero.

1.9 Quadratic forms and sign definitness

Definitions

- Degree of a polynomial
- Form of *n*th degree
- special case: quadratic form

$$Q(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

1.9 Quadratic forms and sign definitness

A quadratic form $Q(x_1, x_2)$ for two variables x_1 and x_2 is defined as

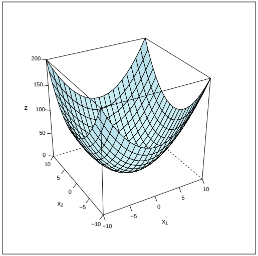
$$Q(x_1, x_2) = \underset{(1\times2)(2\times2)(2\times1)}{\mathbf{x}' \; \mathbf{A} \; \mathbf{x}} = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} \, x_i \, x_j$$

where $a_{ij} = a_{ji}$ and, thus,

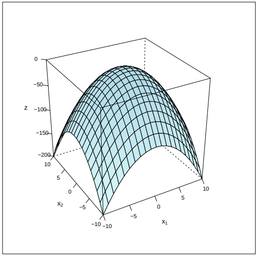
with the symmetric coefficient matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$

1.9 Quadratic forms and sign definitness

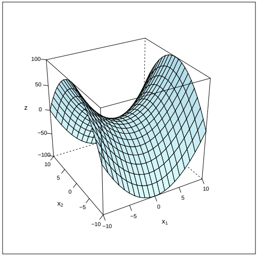
Graph of the positive definite form $Q(x_1, x_2) = x_1^2 + x_2^2$



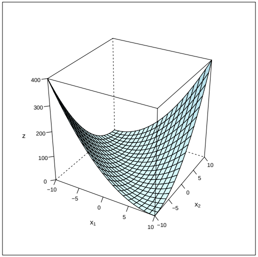
Graph of the negative definite form $Q(x_1, x_2) = -x_1^2 - x_2^2$



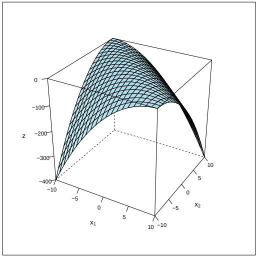
Graph of the indefinite form $Q(x_1, x_2) = x_1^2 - x_2^2$



Graph of the positive semidefinite form $Q(x_1, x_2) = (x_1 + x_2)^2$



Graph of the negative semidefinite form $Q(x_1, x_2) = -(x_1 + x_2)^2$



1.9 Quadratic forms and sign definitness

The quadratic form associated with the matrix $\mathbf{A}$ (and thus the matrix $\mathbf{A}$ itself) is said to be

positive definite,	if $Q = \mathbf{x}'\mathbf{A}\mathbf{x} > \mathbf{0}$	for all $\mathbf{x} \neq \mathbf{0}$
positive semi-definite,	if $Q = \mathbf{x}'\mathbf{A}\mathbf{x} \geq \mathbf{0}$	for all $\mathbf{x}$
negative definite,	if $Q = \mathbf{x}'\mathbf{A}\mathbf{x} < \mathbf{0}$	for all $\mathbf{x} \neq \mathbf{0}$
negative semi-definite,	if $Q = \mathbf{x}'\mathbf{A}\mathbf{x} \leq \mathbf{0}$	for all $\mathbf{x}$

Otherwise the quadratic form is **indefinite**.

Note: For any quadratic matrix $\mathbf{A}$ it holds that $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{B}\mathbf{x}$ with $\mathbf{B} = 0,5 \cdot (\mathbf{A} + \mathbf{A}')$ symmetric.

1.9 Quadratic forms and sign definitness

The quadratic form $Q(\boldsymbol{x})$ is

- positive (negative) definite, if **all** eigenvalues of the matrix $\mathbf{A}$ are positive (negative): $\lambda_j > 0$ ($\lambda_j < 0$) $\forall j = 1, 2, \ldots, n$;
- positive (negative) semi-definite, if **all** eigenvalues of the matrix $\mathbf{A}$ are non-negative (non-positive): $\lambda_j \geq 0$ ($\lambda_j \leq 0$) $\forall j = 1, 2, \ldots, n$ and **at least one** eigenvalue is equal to zero;
- indefinite, if two eigenvalues have different signs.

1.9 Quadratic forms and sign definitness

Properties of positive definite and positive semi-definite matrices

1) Diagonal elements of a positive definite matrix are strictly positive. Diagonal elements of a positive semi-definite matrix are nonnegative.
2) If $\mathbf{A}$ is positive definite, then $\mathbf{A}^{-1}$ exists and is positive definite.
3) If $\mathbf{X}$ is $n \times k$, then $\mathbf{X}'\mathbf{X}$ and $\mathbf{X}\mathbf{X}'$ are positive semi-definite.
4) If $\mathbf{X}$ is $n \times k$ and $\text{rk}(\mathbf{X}) = \mathbf{k}$, then $\mathbf{X}'\mathbf{X}$ is positive definite (and therefore non-singular).