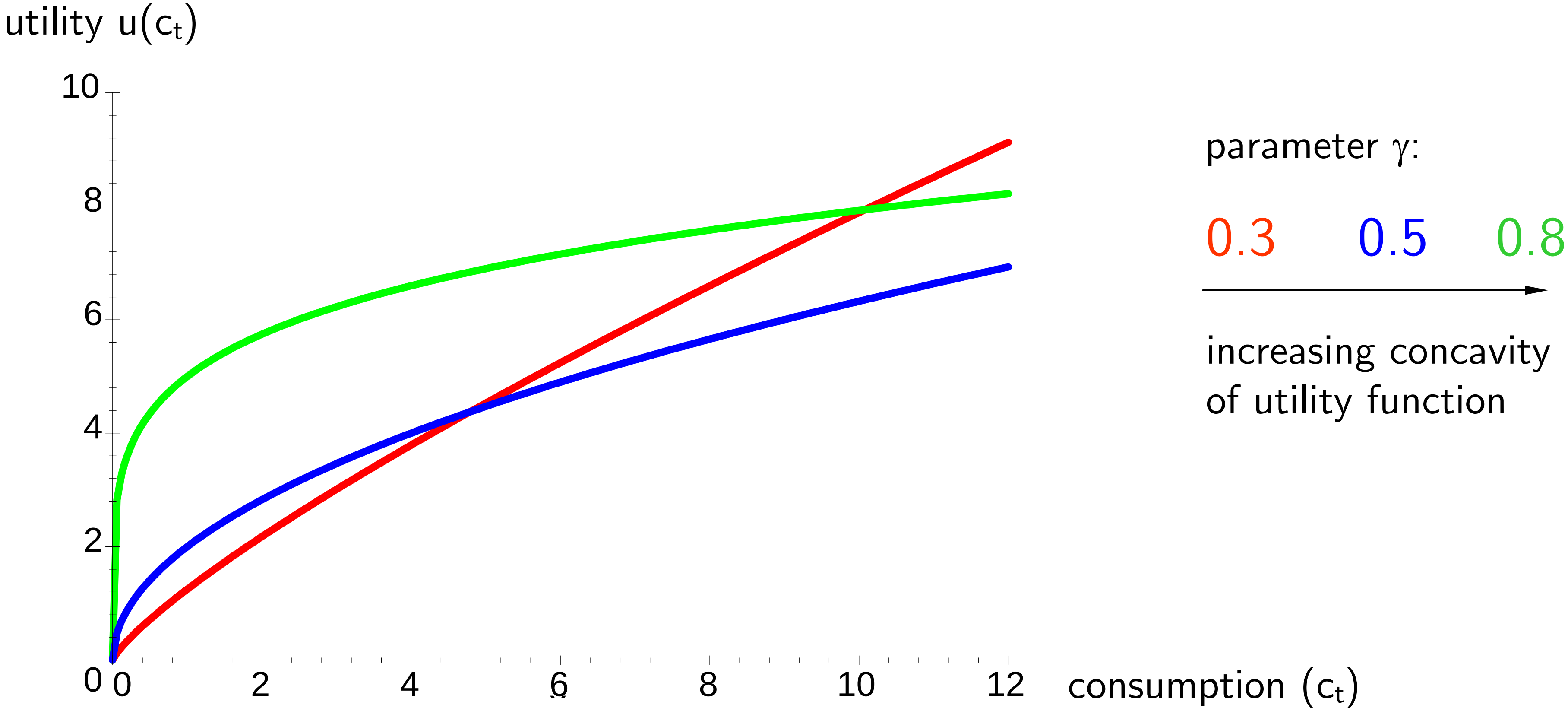
We often use a convenient power utility function (1)

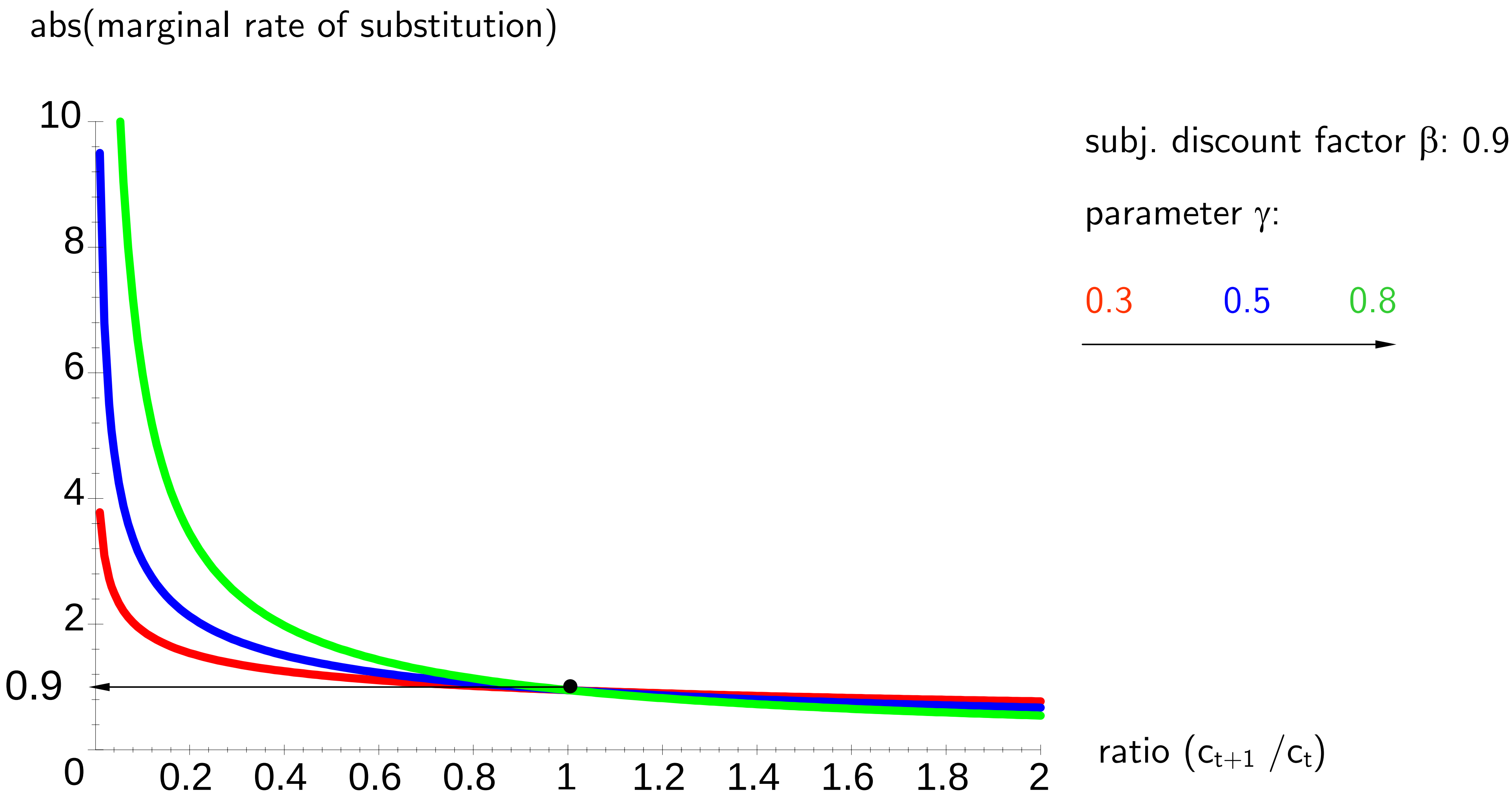
$$u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma} \qquad \lim_{\gamma \to 1} \left(\frac{1}{1-\gamma} c_t^{1-\gamma} \right) = \ln(c_t)$$

$$u'(c_t) = c_t^{-\gamma} \qquad \frac{dc_t}{dc_{t+1}} = \frac{\beta u'(c_{t+1})}{u'(c_t)} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}$$

← Negative of marginal rate of substitution



We often use a convenient power utility function (2)



Measuring absolute risk aversion (1)

Find the risk premium π_Z, such that investor is indifferent to receiving the risk Z and receiving the non-random amount $\mathbb{E}(Z) - \pi_Z$

wealth (non-random) → ; expected value of risky payment → ; expected utility →

$$u\left\{w + \mathbb{E}(Z) - \pi_Z(w)\right\} = \mathbb{E}\left[u(w+Z)\right]$$

risk premium → $\pi_Z(w)$

Assumptions: $$\mathbb{E}(Z) = \mu_Z = 0,\ \sigma_Z^2 \to 0,\ \mathbb{E}\left[\left|(Z-\mu_Z)^3\right|\right] = o\left(\sigma_Z^2\right)$$

neutral risk → $\mu_Z = 0$

Taylor expansion of the left hand side:

$$\begin{aligned} u(w - \pi_Z) &= u(w) - \pi_Z \cdot u'(w) + \frac{1}{2!}\pi_Z^2 \cdot u''(w) + \ldots \\ &= u(w) - \pi_Z \cdot u'(w) + O\left(\pi_Z^2\right) \end{aligned}$$

Measuring absolute risk aversion (2)

Taylor expansion of the right hand side:

$$\begin{aligned}\mathbb{E}\left[u\left(w+Z\right)\right] &= \mathbb{E}\left[u\left(w\right)+Z\cdot u'\left(w\right)+\frac{1}{2!}Z^2u''\left(w\right)+O\left(Z^3\right)\right]\\ &= u\left(w\right)+\mathbb{E}\left(Z\right)\cdot u'\left(w\right)+\frac{1}{2}\sigma_Z^2u''\left(w\right)+o\left(\sigma_Z^2\right)\\ &= u\left(w\right)+\frac{1}{2}\sigma_Z^2u''\left(w\right)+o\left(\sigma_Z^2\right)\end{aligned}$$

Equating both sides:

$$u\left(w\right)-\pi_Z\cdot u'\left(w\right)=u\left(w\right)+\frac{1}{2}\sigma_Z^2u''\left(w\right)+o\left(\sigma_Z^2\right)$$

Risk premium $\longrightarrow$ $$\pi_Z=\frac{1}{2}\sigma_Z^2\cdot r\left(w\right)+o\left(\sigma_Z^2\right)$$

Arrow-Pratt measure of absolute risk aversion $\longrightarrow$ $$r\left(w\right)\triangleq-\frac{u''\left(w\right)}{u'\left(w\right)}=-\frac{d}{dw}\log\left(u'\left(w\right)\right)$$

the more concave the utility function, the higher the risk aversion

Measuring relative (or proportional) risk aversion (1)

Find the proportional risk premium π^*_Z, such that investor is indifferent to receiving the proportional risk $w \cdot Z$ and receiving the non-random amount $\mathbb{E}(w \cdot Z) - w \cdot \pi^*_Z$

$$u\left\{w + \mathbb{E}(w \cdot Z) - w \cdot \pi^*_Z(w)\right\} = \mathbb{E}\left[u(w + w \cdot Z)\right]$$

as above: $\mathbb{E}(Z) = \mu_Z = 0,\ \sigma^2_Z \to 0,\ \mathbb{E}\left[\left|(Z - \mu_Z)^3\right|\right] = o\left(\sigma^2_Z\right)$

Taylor expansion of the left hand side:

$$\begin{aligned} u\left(w - w \cdot \pi^*_Z\right) &= u(w) - w\pi^*_Z \cdot u'(w) + \frac{1}{2!} w^2 \pi^{*2}_Z \cdot u''(w) + \ldots \\ &= u(w) - w\pi^*_Z \cdot u'(w) + O\left(w^2 \pi^{*2}_Z\right) \end{aligned}$$

Measuring relative (proportional) risk aversion (2)

Taylor expansion of the right hand side:

$$\begin{aligned}\mathbb{E}\left[u\left(w+wZ\right)\right] &= \mathbb{E}\left[u\left(w\right)+wZ\cdot u'\left(w\right)+\frac{1}{2!}w^2Z^2u''\left(w\right)+O\left(Z^3\right)\right]\\ &= u\left(w\right)+w\mathbb{E}\left(Z\right)\cdot u'\left(w\right)+\frac{1}{2}w^2\sigma_Z^2u''\left(w\right)+o\left(\sigma_Z^2\right)\\ &= u\left(w\right)+\frac{1}{2}w^2\sigma_Z^2u''\left(w\right)+o\left(\sigma_Z^2\right)\end{aligned}$$

Equating both sides:

$$u\left(w\right)-w\pi_Z^*\cdot u'\left(w\right) = u\left(w\right)+\frac{1}{2}w^2\sigma_Z^2u''\left(w\right)$$

Risk premium ⟶ $$\pi_Z^*\left(w\right) = \frac{1}{2}\sigma_Z^2\cdot r^*\left(w\right)+o\left(\sigma_Z^2\right)$$

Arrow-Pratt measure of proportional risk aversion ⟶ $$r^*\left(w\right)=-w\frac{u''\left(w\right)}{u'\left(w\right)}=w\cdot r\left(w\right)$$

The specification of the utility function implies a close link between risk aversion and intertemporal elasticity of substitution

$$-\frac{u''(c_t)}{u'(c_t)} = \frac{\gamma}{c_t}$$ ← absolute risk aversion coefficient

$$-\frac{c_t \cdot u''(c_t)}{u'(c_t)} = \gamma$$ ← relative risk aversion coefficient

intertemporal elasticity of substitution →

$$\sigma \equiv -\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}} : \frac{d\text{MRS}}{\text{MRS}}$$

$$\frac{1}{\sigma} = -\frac{d\text{MRS}}{d\left(\frac{c_{t+1}}{c_t}\right)} \cdot \frac{\left(\frac{c_{t+1}}{c_t}\right)}{\text{MRS}} = -\frac{-\gamma \cdot \beta \cdot \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma-1} \cdot \left(\frac{c_{t+1}}{c_t}\right)}{\beta \cdot \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}}$$

$$\sigma = \frac{1}{\gamma}$$