Part 3

Turning to explicit mathematics

Point of departure

Systems of explicit mathematics introduced by S. Feferman in 1975. Since then they play an important role in foundational discussions:

- Original aim: formal framework for constructive mathematics, in particular Bishop-style constructive mathematics.
- First vesions of explicit mathematics based on intuitionistic logic; later formulated in a classical framework.
- Close relationship to systems of second order arithmetic and set theory; instrumental for reductive proof theory.
- Logical foundations of functional and object oriented programming languages.

Feferman's three classsic papers:

- A language and axioms for explicit mathematics, in: J. N. Crossley (ed.), Algebra and Logic, Lecture Notes in Mathematics 450, Springer, 1975;
- Recursion theory and set theory: a marriage of convenience, in: J. E. Fenstad, R. O. Gandy, G. E. Sacks (eds.), Generalized Recursion Theory II, Studies in Logic and the Foundations of Mathematics 94, Elsevier, 1978;
- Constructive theories of functions and classes, in: M. Boffa, D. van Dalen,K. McAloon (eds.). Logic Colloquium '78, Studies in Logic and the Foundations of Mathematics 97, Elsevier, 1979.

Basic ontology (modern approach)

Formulated in a language $\mathbb{L}$ with first and second order variables and constants.

The general universe (first order objects)

- Unspecified general objects, (constructive) operations, bitstrings, programs, ...
- These objects form a partial combinatory algebra.

Classes (second order objects)

- Classes are simply collections of objects.
- These classes help to "structure" the universe.
- As we will see, more versatile than "traditional" type theories.

The element relation $\in$ and the naming relation $\Re$

$$t \in S \quad ::: \quad \text{object } t \text{ is an element of class } S$$

(Strong form of polymorphism: an object may belong to many classes.)

Equality of classes defined by

$$S = T \ := \ \forall x(x \in S \leftrightarrow x \in T).$$

Classes can be addressed via there names:

$$\Re(t, S) \quad ::: \quad \text{object } t \text{ is a name of class } S.$$

Explicit representation and equality

(E1) $\exists x \Re(x, S)$,

(E2) $\Re(r, S) \wedge \Re(r, T) \rightarrow S = T$,

(E3) $\Re(r, S) \wedge S = T \rightarrow \Re(r, T)$.

Some abbreviations:

$$s \dot{\in} t := \exists X(\Re(t, X) \wedge s \in X),$$

$$s \doteq t := \exists X(\Re(s, X) \wedge \Re(t, X)),$$

$$S \subseteq T := (\forall x \in S)(x \in T),$$

$$s \dot{\subseteq} t := \exists X, Y(\Re(s, X) \wedge \Re(t, Y) \wedge X \subseteq Y),$$

$$s \in \Re := \exists X \Re(s, X) \quad \text{(although } \Re \text{ is in general not a class).}$$

Basic characteristics of this operational framework

- Reconcile the intensional with the extensional point of view: Intensionality on the level of objects (names) and extensionality on the level of classes.
- The general universe of discourse simply is a patial combinatory algebra; typical examples: Kleene's first and second model, the graph model, the (total) term model, ...
- Self-application of objects – we often call them *operations* – to each other is possible; however, it does not necessarily produce a value. The exact nature of these operations is purposely left open.
- The universe is open-ended but has some simple closure properties.
- **No specific ideology**.

The language $\mathbb{L}$

Basic vocabulary:

- Variables for individuals: $a, b, c, f, g, h, x, y, z, \dots$.
- Variables for classes $A, B, C, R, S, T, X, Y, Z, \dots$.
- Many individual constants and a class constant N.
- Function symbol $\circ$ for (partial) term application.
- Relation symbols $\downarrow$, $\in$, $=$, and $\Re$.

Indiividual terms (r,s,t,...):

$$\text{ind. variables} \mid \text{ind. constants} \mid (s \circ t)$$

As usual:

$$st \;:=\; (s \circ t)$$

$$s_1(s_2 \dots s_n) \;:=\; s_1 s_2 \dots s_n \;:=\; (\dots(s_1 s_2) \dots s_n).$$

Logic of partial terms (Beeson)

$$t\downarrow \;:::\; \text{term } t \text{ has a value;}$$

$$s \simeq t \;:=\; (s\downarrow \vee\, t\downarrow \;\rightarrow\; s = t).$$

Some characteristic properties

- $x\downarrow$.
- $c\downarrow$ if c is a constant.
- $st\downarrow \;\rightarrow\; (s\downarrow \wedge\, t\downarrow)$.
- $A[t] \;\rightarrow\; t\downarrow$ for atomic $A[t]$.
- $A[t] \wedge t\downarrow \;\rightarrow\; \exists x A[x]$.

Moses Schönfinkel (1889 – 1942)

The inventor of *Combinatory Logic*:
Equivalent to predicate logic.

Haskell Brooks Curry (1900 – 1982)

Further development of combinatroy logic.
Mathematical analysis of substitution (and conversion) of terms.
Curry's paradox.

Partial combinatory algebra

Combinatory axioms, pairing and projections

(PCA1) $\mathsf{k} \neq \mathsf{s}$.

(PCA2) $\mathsf{k}ab = a$.

(PCA3) $\mathsf{s}ab\downarrow \;\wedge\; \mathsf{s}abc \simeq (ac)(bc)$.

(PCA4) $\mathsf{p}_0\langle a, b\rangle = a \;\wedge\; \mathsf{p}_1\langle a, b\rangle = b$, where $\langle a, b\rangle := \mathsf{p}ab$.

Immediate consequences

λ-abstraction, fixed point theorem.

A "computational engine", acting on our universe.

λ-Abstraction

For each term t and each variable x we can find a term – written $(\lambda x.t)$ – such that its variables are those of t minus x and

- $(\lambda x.t)\downarrow$ and $(\lambda x.t)x \simeq t$.
- $s\downarrow \to (\lambda x.t)s \simeq t[s/x]$.

Proof.

Induction on t.

(1) If t is the variable x, then $(\lambda x.t) := \mathsf{skk}$.

(2) If t is a variable different from x or a constant, then $(\lambda x.t) := \mathsf{k}t$.

(3) If t is the term $t_1 t_2$, then $(\lambda x.t) := \mathsf{s}(\lambda x.t_1)(\lambda x.t_2)$.

□

Fixed point theorem

There exists a closed term fix such that

$$\mathsf{fix}(f)\downarrow \quad \wedge \quad g = \mathit{fix}(f) \;\rightarrow\; \forall x(gx \simeq fgx).$$

A formula A is called

- *stratified* iff the relationsymbol $\Re$ does not occur in A;
- *elementary* iff it is stratified and does quantify over classes.

Finite axiomatization of uniform elementary comprehension such that:

Theorem

For every elementary formula $\varphi[u, \vec{v}, \vec{W}]$ with at most the indicated free variables there exists a closed term t_φ such that:

1. $\vec{z} \in \Re \;\rightarrow\; t_\varphi(\vec{y}, \vec{z}) \in \Re$,
2. $\Re(\vec{z}, \vec{Z}) \;\rightarrow\; \forall x(x \mathrel{\dot{\in}} t_\varphi(\vec{y}, \vec{z}) \;\leftrightarrow\; \varphi[x, \vec{y}, \vec{Z}])$.

Hence, $t_\varphi(\vec{y}, \vec{z})$ is a name of $\{x : \varphi[x, \vec{y}, \vec{Z}]\}$.

Comprehension for non-stratified formulas may lead to inconsisteny.

The natural numbers $(\mathsf{N}, 0, \mathsf{s_N}, \mathsf{p_N}, \mathsf{d_N}, \mathsf{r_N})$

Some abbreviations:

$$f \in (\mathsf{N}^k \to \mathsf{N}) \;:=\; (\forall x_1, \ldots, x_k \in \mathsf{N})(f(x_1, \ldots x_k) \in \mathsf{N})),$$
$$t' \;:=\; \mathsf{s_N} t.$$

Basic N-axioms

(N1) $0 \in \mathsf{N} \;\wedge\; a \in \mathsf{N} \;\to\; a' \in \mathsf{N}.$

(N2) $a' \neq 0 \;\wedge\; \mathsf{p_N} 0 = 0 \;\wedge\; \mathsf{p_N}(a') = a.$

(N3) $x \in \mathsf{N} \wedge y \in \mathsf{N} \wedge x = y \;\to\; \mathsf{d_N}(a, b, x, y) = a.$

(N4) $x \in \mathsf{N} \wedge y \in \mathsf{N} \wedge x \neq y \;\to\; \mathsf{d_N}(a, b, x, y) = b.$

Number-valued primitive recursion

(N5) $a \in \mathsf{N} \wedge f \in (\mathsf{N}^2 \to \mathsf{N}) \;\to\; \mathsf{r_N}(a, f) \in (\mathsf{N} \to \mathsf{N})$.

(N6) $\left.\begin{array}{c} a, b \in \mathsf{N} \wedge f \in (\mathsf{N}^2 \to \mathsf{N}) \\ \wedge\; g = \mathsf{r_N}(a, f) \end{array}\right\} \to\; g0 = a \wedge g(b') = f(b, (gb))$.

The elementary theoy of classes EC is formulated in the classical logic of partial individual terms with equality.

Elementary theory of classes

$$\mathsf{EC} \quad := \quad (\mathrm{E}) + (\mathrm{PCA}) + (\mathrm{N}) + (\mathrm{el.comp.})$$

What are sets of natural numbers?

$$Set_{\mathsf{N}} := \{f : f \in (\mathsf{N} \to \mathsf{N})\}$$
$$b \,\varepsilon\, a := a \in Set_{\mathsf{N}} \wedge b \in \mathsf{N} \wedge ab = 0.$$

So – provisionally – we assume that *sets of natural numbers* are represented by total operations from N to N.

Forms of induction on N

Set induction (S-$\mathsf{I_N}$)

$$0 \,\varepsilon\, a \wedge (\forall x \in N)(x \,\varepsilon\, a \to x' \,\varepsilon\, a) \;\to\; (\forall x \in N)(x \,\varepsilon\, a).$$

Class induction (C-$\mathsf{I_N}$)

$$0 \in S \wedge (\forall x \in N)(x \in S \to x' \in S) \;\to\; (\forall x \in N)(x \in S).$$

Formula induction ($\mathbb{L}$-$\mathsf{I_N}$)

$$\varphi[0] \wedge (\forall x \in N)(\varphi[x] \to \varphi[x']) \;\to\; (\forall x \in N)\varphi[x].$$

Theorem (First observation)

1. $\mathsf{EC} + (\mathsf{S}\text{-}\mathsf{I_N}) \equiv \mathsf{PRA}$.
2. $\mathsf{EC} + (\mathsf{C}\text{-}\mathsf{I_N}) \equiv \mathsf{ACA_0} \equiv \mathsf{PA}$.
3. $\mathsf{EC} + (\mathbb{L}\text{-}\mathsf{I_N}) \equiv \mathsf{ACA}$.

Remark

- With set induction we can prove that sets have specific prioperties. In general, this form of induction cannnot be used in order to show that specific terms represent sets.
- If only set induction is available, axioms (PR1) and (PR2) are needed for having closure under primitive recursion.
- If class induction is available, the usual fixed point theorem of a PCA proves the existence of a closed term with the properties of $\mathsf{r_N}$.

Lemma (Set-valued recursion)

There exists a closed term rec *such that* $\mathsf{EC} + (\mathsf{C}\text{-}\mathsf{I_N})$ *proves:*

$$a \in Set_{\mathsf{N}} \;\wedge\; b \in \mathsf{N} \;\wedge\; f \in (\mathsf{N} \times Set_{\mathsf{N}} \to Set_{\mathsf{N}}) \;\wedge\; g = \mathsf{rec}(a, f) \;\;\to$$

$$g \in (\mathsf{N} \to Set_{\mathsf{N}}) \;\wedge\; g(0) = a \;\wedge\; g(b') = f(b, g(b)).$$

But thus far, Set_{N} has very weak closure properties. For example, it is not closed under arithmetical comprehension.

The unbounded minimum (or search) operator μ

$(\mu.1)$ $\quad f \in (\mathsf{N} \to \mathsf{N}) \;\leftrightarrow\; \mu f \in \mathsf{N}$,

$(\mu.2)$ $\quad f \in (\mathsf{N} \to \mathsf{N}) \wedge (\exists x \in \mathsf{N})(fx = 0) \;\to\; f(\mu f) = 0.$

Remark

Least standard model of the applicative part of $\mathsf{EC}(\mu)$ by means of Π^1_1 recursion theory: The objects are the natural numbers and

$$(x \circ y) \quad \text{interpreted as} \quad \{x\}^{\mathsf{E}}(y),$$

where E is the well-known type-2 equality functional

$$\mathsf{E}(h) \;=\; \begin{cases} 0 & \text{if } \exists n(h(n) = 0), \\ 1 & \text{otherwise.} \end{cases}$$

Theorem

1. $\mathsf{EC}(\mu) + (\mathsf{S}\text{-}\mathsf{I_N}) \;\equiv\; \mathsf{PA}.$
2. $\mathsf{EC}(\mu) + (\mathsf{C}\text{-}\mathsf{I_N}) \;\equiv\; \Pi^0_\infty\text{-}\mathsf{CA}_{<\varepsilon_0} \;\equiv\; \Sigma^1_1\text{-}\mathsf{AC}.$
3. $\mathsf{EC}(\mu) + (\mathbb{L}\text{-}\mathsf{I_N}) \;\equiv\; \Pi^0_\infty\text{-}\mathsf{CA}_{<\varepsilon_{\varepsilon_0}}.$

Remark

- $\mathsf{EC}(\mu) + (\mathsf{S\text{-}I_N})$ may be considered as a reformulation of Feferman's system W and is proof-theoretically equivalent to his $\mathsf{K}^{(\alpha)}$.
- $\mathsf{EC}(\mu) + (\mathsf{C\text{-}I_N})$ is proof-theoretically equivalent to $\mathsf{K}^{(\beta)}$.

Adding join (J)

If a names a class and f maps all elements of this class to classes,

$$(\forall x \mathrel{\dot{\in}} a)(fx \in \Re),$$

we write $b = \Sigma(a, f)$ for the assertion

$$\forall x(x \mathrel{\dot{\in}} b \;\leftrightarrow\; x = \langle \mathsf{p}_0 x, \mathsf{p}_1 x \rangle \wedge \mathsf{p}_0 x \mathrel{\dot{\in}} a \wedge \mathsf{p}_1 x \mathrel{\dot{\in}} f(\mathsf{p}_0 x)),$$

stating that b names the disjoint union of the classes named fy for y ranging over a. The join axiom (J) claims the existence of such disjoint unions.

Join (J)

$$a \in \Re \wedge (\forall x \mathrel{\dot{\in}} a)(fx \in \Re) \;\rightarrow\; \mathsf{j}(a, f) \in \Re \wedge \mathsf{j}(a, f) = \Sigma(a, f).$$

Theorem

1. $\mathsf{EC} + (\mathsf{J}) + (\mathsf{S}\text{-}\mathsf{I_N}) \;\equiv\; \mathsf{PRA}.$
2. $\mathsf{EC} + (\mathsf{J}) + (\mathsf{C}\text{-}\mathsf{I_N}) \;\equiv\; \mathsf{ACA_0} \;\equiv\; \mathsf{PA}.$
3. $\mathsf{EC} + (\mathsf{J}) + (\mathbb{L}\text{-}\mathsf{I_N}) \;\equiv\; \Pi^0_1\text{-}\mathsf{CA}_{<\varepsilon_0} \;\equiv\; \Sigma^1_1\text{-}\mathsf{AC}.$

Theorem

1. $\mathsf{EC}(\mu) + (\mathsf{J}) + (\mathsf{S}\text{-}\mathsf{I_N}) \;\equiv\; \mathsf{ACA_0} \;\equiv\; \mathsf{PA}.$
2. $\mathsf{EC}(\mu) + (\mathsf{J}) + (\mathsf{C}\text{-}\mathsf{I_N}) \;\equiv\; \Pi^0_1\text{-}\mathsf{CA}_{<\varepsilon_0} \;\equiv\; \Sigma^1_1\text{-}\mathsf{AC}.$
3. $\mathsf{EC}(\mu) + (\mathsf{J}) + (\mathbb{L}\text{-}\mathsf{I_N}) \;\equiv\; \Pi^0_1\text{-}\mathsf{CA}_{<\varphi\varepsilon_0 0}.$

Some ontological observations

Two forms of power classes

- **Strong power class** (SP). For every class X there exists a class Y such that Y consists exactly of the names of all subclasses of X,

$$\forall X \exists Y \forall z (z \in Y \leftrightarrow \exists Z (\Re(z, Z) \wedge Z \subseteq X)).$$

- **Weak power class** (WP). It only claims that for each class X there exists a class Y such that each element of Y names a subclass of X and for any subclass of X at least one of its names belongs to Y,

$$\forall X \exists Y ((\forall z \in Y)(\exists Z \subseteq X)(\Re(z, Z)) \wedge (\forall Z \subseteq X)(\exists z \in Y)\Re(z, Z)).$$

Remark

Even the uniform version of (WP) is consistent with EC.

Theorem

1. *The names of a class never form a class, i.e.*

$$\mathsf{EC} \vdash \forall X \neg \exists Y (Y = \{z : \Re(z, X)\}).$$

2. *Hence,* (SP) *is inconsistent with* EC.

3. *It is consistent with* EC *(though not provable there) to assume that there exists the class of all names.*

4. *The theory* EC + (J) *proves that not all objects are names.*

5. *The theory* EC + (J) *proves the negation of* (WP).

Operational extensionality (Op-Ext)

$$\forall f, g(\forall x(fx \simeq gx) \rightarrow f = g).$$

Full definition by cases (D_V)

$$(a = b \rightarrow d_V(u, v, a, b) = u) \wedge (a \neq b \rightarrow d_V(u, v, a, b) = v).$$

Remark

If we set (Tot) $:= \forall x, y(xy\downarrow)$, then we have:

1. EC + (Op-Ext) + (D_V) is inconsistent.
2. EC + (Op-Ext) + (Tot) is consistent.
3. EC + (Op-Ext) + $\forall x(x \in \mathsf{N})$ is inconsistent.
4. EC + (Tot) + $\forall x(x \in \mathsf{N})$ is inconsistent.

Proof of (1). We set

$$s := \mathsf{fix}(\lambda yx.\mathsf{d}_V(1,0,y,(\lambda z.0)))$$

and thus have

$$\mathsf{s}x \simeq (\lambda yx.\mathsf{d}_V(1,0,y,(\lambda z.0)))sx \simeq \mathsf{d}_V(1,0,s,(\lambda z.0)).$$

Hence, if $s = (\lambda z.0)$, then $\mathsf{s}x = 1$ for all x, which is impossible. Therefore, $s \neq (\lambda z.0)$. Hence, $sx = 0$ for all x. By (Op-Ext) we thus have $s = (\lambda z.0)$. But this is a contradiction.