

Time series properties of asset prices and time series models of heteroskedasticity

Readings:

Hamilton (1994), Chapter 21

In the previous sessions we have focussed on the cross sectional properties of prices and returns

So far: cross sectional differences between assets

$$\mathbb{E}(R^i) - R^f = -R^f Cov(m, R^i)$$

Time series properties of asset prices and returns.  
Especially: **Predictability** of asset returns.

What does theory have to say about predictability?

Practice?

# Time series of asset returns are realizations of stochastic processes

$\{X_t\}_{t=-\infty}^{\infty}$ , a stochastic process, is a sequence of random variables indexed and ordered by time.

$$\left\{ \overbrace{\dots, X_0, X_1, \dots, X_T, \dots}^{\text{a realization of the process}} \right\}$$

observed sample

Analyze distributional properties of sample period  $\hat{=}$  analysis of (multivariate) distribution of random vector.

Note: No (deterministic) chaos!

## Conditional vs unconditional distributions and moments

- joint distribution  $f_{X_t X_{t+1}}(x_t, x_{t+1})$
- marginal distribution  $f_{X_t}(x_t)$
- conditional distribution of  $X_t | X_{t-1} \sim ?$
- unconditional moments  $\mathbb{E}(X_t), \text{Var}(X_t), \dots$
- conditional moments  $\mathbb{E}(X_{t+1} | X_t, X_{t-1}, \dots), \dots$

Unconditional moments  $\neq$  conditional moments  $\Rightarrow$  predictability?

To study the time series properties of asset prices and returns we review some fundamentals of time series analysis

Weak stationarity

$$\left. \begin{aligned} \mathbb{E}(X_t) &= \mu \\ \text{Var}(X_t) &= \sigma^2 \\ \text{Cov}(X_t, X_{t-j}) &= \gamma_j \end{aligned} \right\} \begin{array}{l} \text{unconditional} \\ \text{moments are not} \\ \text{time dependent} \end{array}$$

serial dependence  $\text{Cov}(X_t, X_{t-j}) = \gamma_j \neq 0$  for  $j \neq 0$   
 $\Rightarrow$  predictability

# Martingale Processes

$\mathbb{E}(X_{t+1} | I_t) = X_t$   $I_t$  : information available time  $t$

$\{X_t, X_{t-1}, \dots\} \subset I_t$

$\{X_t\}$  A martingale w.r.t  $I_t$

$\mathbb{E}(X_{t+1} | I_t)$  **best** forecast of  $X_{t+1}$  in terms of

$$MSE = \mathbb{E} \left[ (X_{t+1|t}^* - X_{t+1})^2 \right]$$

Using  $\mathbb{E}(X_{t+1} | I_t)$  for  $X_{t+1|t}^*$  yields smallest MSE

[Proof: Hamilton (1994): Time Series Analysis, page 72f.]

For a martingale: "best" forecast of tomorrow: observed value of process today

## Are asset prices martingales?

$\mathbb{E}(Y_{t+1}|I_t) = 0$  a martingale difference process

$Y_t = X_t - X_{t-1}$   $\mathbb{E}(X_{t+1} - X_t|I_t) = 0$  ( $X_t \in I_t$ )

Future **changes** are not forecastable using past information (do not improve MSE)

### **Hypothesis:**

Do asset prices follow a martingale process  $\Rightarrow$ , i.e. price changes unforecastable?

**Theory?**

**Practice?**

Marginal utility weighted prices follow martingales (in the absence of dividends)

$$\mathbb{E}_t \left( m_{t+1} x_{t+1} \right) = p_t \quad x_{t+1} = p_{t+1} + d_{t+1}$$

$$p_t = \mathbb{E} \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} | I_t \right) \quad \text{compare to } \mathbb{E} \left( X_{t+1} | I_t \right) = X_t$$

Assume  $\beta \approx 1$  and no dividends  $d_{t+1} = 0$

$$\mathbb{E} \left( u'(c_{t+1}) p_{t+1} | I_t \right) = u'(c_t) p_t \quad u'(c_t) p_t \equiv \tilde{p}_t$$

$$\mathbb{E} \left( \tilde{p}_{t+1} | I_t \right) = \tilde{p}_t$$

Marginal utility weighted prices follow a martingale process



In a risk neutral world with no dividends and no time preferences prices follow a martingale. Predictability in the short run?

In a **risk neutral world**  $u'(c_t)$  constant:

$$\mathbb{E}(p_{t+1}|I_t) = p_t$$

**Short run**, high frequency ( e.g. daily)

$$\beta = 1, c_t \text{ almost constant, } \Rightarrow u'(c_t) = u'(c_{t+1})$$

$$\Rightarrow \mathbb{E}(p_{t+1}|I_t) = p_t \text{ in short run!}$$

no better forecast of  $p_{t+1}$  than  $p_t$  in terms of MSE

$$\mathbb{E}_t\left(\frac{p_{t+1}}{p_t}\right) = \mathbb{E}_t(R_{t+1}) = 1 \text{ (coin flips) } \underbrace{\mathbb{E}(R_{t+1} - 1)}_{\text{net return}} = 0$$

**In practice...**

technical analysis, trend lines, resistance lines, double shoulders...

## Predictability in the short run? (1)

### An ARMA model for asset returns?

$$R_{t+1} = c + \phi_1 R_t + \dots + \phi_p R_{t-p+1} + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1} + \dots + \theta_p \varepsilon_{t-p+1} + \varepsilon_{t+1}$$

where

$$\mathbb{E}(\varepsilon_t) = 0, \text{Var}(\varepsilon_t) = \sigma^2, \text{Cov}(\varepsilon_t \varepsilon_{t-j}) = 0 \quad j > 0$$

$$\mathbb{E}(\varepsilon_t | I_{t-1}) = 0$$

A useful model?

$$\mathbb{E}(R_{t+1} | I_t) = c + \phi_1 R_t + \phi_2 R_{t-1} + \dots + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1}$$
$$\{R_t, R_{t-1}, \varepsilon_t, \varepsilon_{t-1}, \dots\} \subset I_t$$

If theory correct  $\phi_1 = \phi_2 = \dots = \theta_1 = \theta_2 = 0$

## Predictability in the short run? (2)

Some specific martingale processes

### **Random Walk**

$$p_t = p_{t-1} + \varepsilon_t \quad \mathbb{E}_t(\varepsilon_t) = 0$$

#### **RW type 1**

$\{\varepsilon_t\}$  i.i.d independent, identically distributed

#### **RW type 2**

$\{\varepsilon_t\}$  independent, but not necessarily identically distributed

#### **RW type 3**

$\{\varepsilon_t\}$  uncorrelated (less restrictive than independence)

Tests for random walk hypothesis of asset prices (Chapter 3 in Campbell/Lo/McKinlay) Only weak evidence for short run predictability of asset returns.

(Microstructure effects: bid/ask bounce)

## Predictability in the long run?

From

$$\mathbb{E}_t(R_{t+1}^i) - R_t^f = -\frac{\text{Cov}_t(m_{t+1}, R_{t+1}^i)}{\mathbb{E}_t(m_{t+1})}$$

using  $m_t = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$  and lognormal consumption growth  $\frac{c_{t+1}}{c_t}$

$$\mathbb{E}_t(R_{t+1}^i) - R_t^f \approx \gamma_t \sigma_t(\Delta \ln c_{t+1}) \sigma_t(R_{t+1}^i) \rho_t(m_{t+1}, R_{t+1}^i)$$

## When setting up a time series model in finance we usually use log returns (1)

It is useful to take 'log returns' (continuously compounded returns)

Use  $r_{t+1} = \ln\left(\frac{p_{t+1}}{p_t}\right)$  instead of  $R_{t+1} = \frac{p_{t+1}}{p_t}$  (gross return)

$$r_{t+1} \approx \frac{p_{t+1} - p_t}{p_t} = \frac{p_{t+1}}{p_t} - 1 = R_{t+1} - 1 \quad (\text{net return})$$

e.g.  $p_{t+1} = 105$     $p_t = 100$   
 $R_{t+1} = 1.05$    net return = 0.05  
 $r_t = 0.049$

Continuous compounding between  $t$  and  $t + 1$

When setting up a time series model in finance we usually use log returns (2)

Distributional assumption for  $R_{t+1} = \frac{p_{t+1}}{p_t}; [0, \infty)$

Normal distribution?  $R_{t+1} - 1; [-1, \infty)$

Assume:  $r_t = \ln\left(\frac{p_{t+1}}{p_t}\right) \sim N(\mu, \sigma^2)$

$\Rightarrow \frac{p_{t+1}}{p_t} = \exp(r_t) \sim \text{lognormal defined on } (0, \infty)$

When setting up a time series model in finance we usually use log returns (3)

Multiperiod returns:  $k > 1$

$$\text{gross returns: } \frac{p_{t+k}}{p_t} = \underbrace{\frac{p_{t+1}}{p_t} \cdot \frac{p_{t+2}}{p_{t+1}} \cdot \frac{p_{t+3}}{p_{t+2}} \cdot \dots \cdot \frac{p_{t+k}}{p_{t+k-1}}}_{\text{multiplicative}}$$

$$\begin{aligned} \text{log returns: } \ln\left(\frac{p_{t+k}}{p_t}\right) &= \underbrace{\ln\left(\frac{p_{t+1}}{p_t}\right) + \ln\left(\frac{p_{t+2}}{p_{t+1}}\right) \dots}_{\text{additive}} \\ &= \ln\left(\frac{p_{t+1}}{p_t} \cdot \frac{p_{t+2}}{p_{t+1}} \cdot \frac{p_{t+3}}{p_{t+2}} \cdot \dots \cdot \frac{p_{t+k}}{p_{t+k-1}}\right) \end{aligned}$$

When setting up a time series model in finance we usually use log returns (4)

(Asymptotic) distribution of sum of (normal) random variables known.

Distribution of product of random variables more difficult, especially asymptotic distribution

LLNs and CLTs exist for sums of random variables



## Stylized facts of financial return data

- low serial correlation in (log) returns (in line with theory, if prices are martingales)
- significant correlation in squared returns

A simple model to account for these stylized facts

$$r_t = c + u_t$$

$$\mathbb{E}(u_t) = 0, \text{Var}(u_t) = \mathbb{E}(u_t^2) = \sigma^2, \mathbb{E}(u_t u_{t-j}) = 0 \text{ for } j \neq 0$$

$\Rightarrow r_t$  and  $u_t$  white noise,  $\underbrace{\mathbb{E}(r_t) = c \text{ and } \text{Var}(r_t) = \sigma^2}_{\text{unconditional}}$

$$\text{Cov}(r_t, r_{t-j}) = 0 \quad \forall j \neq 0$$

The success of Engle's ARCH is due to the fact that the model can take into account the fundamental time series properties of asset prices (1)

For the AutoRegressive Conditional Heteroskedasticity (ARCH) model Engle assumes

$$u_t = \sqrt{h_t} \cdot \varepsilon_t$$

1.  $\varepsilon_t \sim N(0, 1)$  i.i.d.
2.  $\mathbb{E}(\varepsilon_t | I_{t-1}) = 0$  exogenous identical shocks (unpredictable)
3.  $h_t = f(r_{t-1}^2)$  or  $h_t = f(|r_{t-1}|)$  or longer lags of absolute or squared returns. ARCH(1):  $h_t = d + a_1 r_{t-1}^2$

The success of Engle's ARCH is due to the fact that the model can take into account the fundamental time series properties of asset prices (2)

For the ARCH(1) specification

$$h_t = d + a_1 r_{t-1}^2$$

$$\begin{aligned}\mathbb{E}_t(r_t) &= c + \sqrt{h_t} \cdot \mathbb{E}_t(\varepsilon_t) \\ &= c + \sqrt{d + a_1 r_{t-1}^2} \cdot 0 = c\end{aligned}$$

$$\begin{aligned}Var_t(r_t) &= Var(c) + (d + a_1 r_{t-1}^2) Var_t(\varepsilon_t) \\ &= d + a_1 r_{t-1}^2 \\ &= h_t \quad (\text{conditional variance, } \sqrt{h_t} \text{ conditional "volatility"})\end{aligned}$$

Remark: Volatility sometimes defined as **annualized** (log) return standard deviation. With  $\sigma_t = \sqrt{h_t}$  the standard deviation of daily log returns we annualize  $\sigma_{ann.} = T \cdot \sigma_t$  ( $T$  number of trading days)

(ML) estimated coefficient  $a_1$  significantly different from zero (positive)!

$\Rightarrow$  variance of return in  $t+1$  predictable given time  $t$  information!

## A plethora of conditional volatility models have been proposed

- asymmetric responses of return variance to positive or negative return shock?
- persistence of shocks (ARCH) only one lag period or longer?  
→ GARCH:  $h_t = d + \sum_{i=1}^q \alpha_i r_{t-i}^2 + \sum_{j=1}^p \beta_j h_{t-j}$
- Long memory, fractionally integrated GARCH
- shocks  $\varepsilon$  normally distributed?  
fat tails, skewness (large negative shocks more likely)
- How to ensure non-negativity of conditional variance  $h_t$ ?

## A plethora of model variants has been proposed

- ARCH-in-Mean  $r_t = d + \delta h_{t-1} + \sqrt{h_t} \varepsilon_t$
- multivariate extensions  $\Rightarrow$  multivariate ARCH: Conditional covariances of asset returns (correlations) predictable.

# A successful model: Nelson's E-ARCH model (1)

## Nelson's Exponential ARCH

Standard assumptions:

$$\begin{aligned}r_t &= \mu + u_t \\u_t &= \sqrt{h_t}v_t\end{aligned}$$

where

$$E_t(v_t) = 0$$

$$E_t(v_t^2) = 1$$

$$\{v_t\} \text{ iid, i.e. } \text{Var}_t(r_t) = h_t$$

$r_t$  a white noise process as above.

However, log of conditional variance evolves as:

$$\ln h_t = \zeta + \pi\{|v_{t-1}| - \mathbb{E}(|v_{t-1}|) + \aleph v_{t-1}\}$$

## A successful model: Nelson's E-ARCH model (2)

$$\ln h_t = \zeta + \pi\{|v_{t-1}| - \mathbb{E}(|v_{t-1}|)\} + \aleph v_{t-1}$$

Non-negativity of  $h_t = \text{Var}_t(r_t)$  guaranteed .

Asymmetric effects positive and negative return shocks possible:

- $\pi > 0 \rightarrow$  deviation of absolute iid shock  $|v_{t-1}|$  from expectation  $\mathbb{E}(|v_{t-1}|)$  increases volatility forecast (c.p.)
- $-1 < \aleph < 0$  positive return shock  $v_{t-1} > 0$  increases volatility forecast  $h_{t+1}$  less than negative return shock  $v_{t-1} < 0$
- $\aleph < -1$  positive return shock  $v_{t-1} > 0$  c.p. decreases volatility forecast  $h_{t+1}$ .

## A successful model: Nelson's E-ARCH model (3)

$$\ln h_t = \zeta + \pi\{|v_{t-1}| - \mathbb{E}(|v_{t-1}|) + \aleph v_{t-1}\}$$

Economic explanation (heuristic) for  $-1 < \aleph < 0$ :

Leverage effect

stock prices  $\downarrow \Rightarrow$  value of ratio  $\frac{\text{value equity}}{\text{corporate dept}} \downarrow \Rightarrow$   
risk of holding stocks increases.

Note: Extendable to EGARCH model (lagged  $\ln h_{t-j}$  on right hand side)



## Uses of ARCH type models

- Forecast return variances and covariances for VaR models
- volatility forecast to feed in Black/Scholes formula (practitioners approach)
- Estimate and forecast time varying beta

$$\beta_{it} = \frac{Cov_t(R_t^m, R_t^i)}{Var_t(R_t^m)} \Rightarrow \text{asset pricing}$$

Modelling evolution of conditional covariance in same fashion: Bivariate ARCH models

- Portfolio selection: forecast variance-covariance matrix of assets in portfolio (multivariate ARCH models)