Advanced Mathematical Methods

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5 Mathematical Statistics

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Outline: Mathematical Statistics

5.5 Specific probability distributions
5.6 Distribution of a function of a random variable
5.7 Moment generating functions (MGF)

Readings

- A. Papoulis and A. U. Pillai. *Probability, Random Variables and Stochastic Processes.*
  Mc Graw Hill, fourth edition, 2002, Chapter 5

Online References

MIT Course on Probabilistic Systems Analysis and Applied Probability (by John Tsitsiklis)

- Discrete RVs II: Functions of RV, conditional probabilities, specific distribution, total expectation theorem, joint probabilities
  https://www.youtube.com/watch?v=-qCEoqpwjf4

5.5 Specific probability distributions

Some distributions stem from experimental situations.

Existence theorem

For $F_X(x)$ to be a distribution function, it must hold that

(i) $F_X(x) = \int\limits_{-\infty}^{x} f(u)du$

(ii) $f(x)$ non-negative and $\int\limits_{-\infty}^{\infty} f(x)dx = 1$

(iii) $F_X(x)$ continuous from the right and

(iv) monotonically increasing from 0 to 1 as x goes from $-\infty$ to ∞

4.5 Specific probability distributions

The normal distribution

X is a gaussian or normal random variable with parameters μ and σ^2 if its density function is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

denoted $X \sim N(\mu, \sigma^2)$

linear transformation is also normally distributed:
if $X \sim N(\mu, \sigma^2)$, then $a + bX \sim N(a + b\mu, b^2\sigma^2)$

4.5 Specific probability distributions

standardization of X leads to standard normal distribution:

$$a = -\frac{\mu}{\sigma} \quad , \quad b = \frac{1}{\sigma}$$

$$z = \frac{x - \mu}{\sigma} \sim N(0, 1)$$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$

Thus , if $X \sim N(\mu, \sigma)$, then $f(x) = \frac{1}{\sigma}\Phi\left(\frac{x-\mu}{\sigma}\right)$

4.5 Specific probability distributions

The χ^2 distribution:

X is said to be $\chi^2(n)$ with n degrees of freedom if

$$f_X(x) = \begin{cases} \frac{x^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} e^{-\frac{x}{2}} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

if $z \sim N(0,1)$, then $x = z^2 \sim \chi^2(1)$

if z_i are **iid** $N(0,1)$, then $\sum_{i=1}^{n} z_i^2 \sim \chi^2(n)$

5.6 Distribution of a function of a random variable

transformation of the random variable X to a new random variable Y using a measurable function $g(\cdot)$:

$$Y = g(X)$$

requirements:

- $g(\cdot)$ needs to be invertible (monotonic function)
- $g(\cdot)$ needs to be continuously differentiable

5.6 Distribution of a function of a random variable

Transformation theorem

X is continuous with pdf $f_X(x)$. $Y = g(X)$ is strictly monotonous and continuously differentiable,

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot |\frac{d}{dy}(g^{-1}(y))| & \text{for} \quad y \in \{y : y = g(x); x \in \mathbb{W}_X\} \\ 0 & \text{else} \end{cases}$$

$\mathbb{W}_X$ is the domain of X

5.7 Moment generating functions (MGF)

for a random variable X with pdf $f_X(x)$, the MGF is

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \begin{cases} \sum\limits_i e^{tx_i} f_X(x_i) & \text{if } X \text{ is discrete} \\ \int\limits_{-\infty}^{\infty} e^{tx} f_X(x) dx & \text{if } X \text{ is continuous} \end{cases}
\end{aligned}
$$

if the MGF exists, the $k-th$ uncentered moment of X is given as

$$M_X^{(k)}(0) = \left.\frac{d^k M_X(t)}{dt^k}\right|_{t=0} = \mu'_k = E[X^k]$$

5.7 Moment generating functions (MGF)

if $M_X(t)$ exists, then the MGF of $Y = a + bX$ is

$$\begin{aligned} M_Y(t) &= E[e^{t(aX+b)}] \\ &= e^{tb} \cdot E[e^{t(aX)}] \\ &= e^{tb} \cdot E[e^{(ta)X}] \\ &= e^{tb} \cdot M_X(at) \end{aligned}$$

if X and Y are independent, then the MGF of $X + Y$ is $M_X(t) \cdot M_Y(t)$

Note: the MGF of sums of random variables does not always exist