Advanced Mathematical Methods

WS 2022/23

3 Integral calculus

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WIRTSCHAFTS- UND SOZIALWISSENSCHAFTLICHE FAKULTÄT

Outline: Integral calculus

3.1 Indefinite integrals
3.2 Rules of integration
3.3 Definite integrals
3.4 Leibniz's Formula
3.5 Improper integrals
3.6 Double integrals

Readings

- Knut Sydsaeter and Peter Hammond. *Essential Mathematics for Economic Analysis.*
  Prentice Hall, third edition, 2008, Chapter 9
- Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne Strøm. *Further Mathematics for Economic Analysis.*
  Prentice Hall, 2008, Chapter 4

Online References

MIT course on Single Variable Calculus (David Jerison)

- Lecture 15: Antiderivatives
- Lecture 18: Definite Integrals
- Lecture 19: First Fundamental Theorem
- Lecture 20: Second Fundamental Theorem
- Lecture 30: Integration by Parts
- Lecture 36: Improper Integrals
- Lecture 37: Infinite Series

3.1 Indefinite integrals

Definition: Indefinite Integral

A differentiable function $F(x)$ is the ***indefinite integral*** or ***antiderivative*** of $f(x)$ if $F'(x) = f(x)$:

$$F(x) = \int f(x)dx$$

$f(x)$ is the *integrand* and x the *variable of integration*.

Note: The function $F(x)$ is not unique: Let $F(x)$ be the indefinite integral of $f(x)$. For any constant $C \in \mathbb{R}$, $F(x) + C$ is an indefinite integral of $f(x)$ as well.

As $F(x) + C$ is not to be regarded as one definite function, but as a whole class of functions, all having the same derivative f, the integral is called an *indefinite* integral.

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3.2 Rules of Integration

Basic rules

1. $\int 1 \cdot dx = \int dx = x + C$
2. $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1$
3. $\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C, \qquad x \neq 0$
4. $\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C, \qquad a \neq 0$
5. $\int a^x \, dx = \frac{a^x}{\ln(a)} + C, \qquad a > 0 \text{ and } a \neq 1$

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3.2 Rules of Integration

Assume that for $f(x)$ and $g(x)$ the domain is limited as necessary.

Constant Factor

$$\int a \cdot f(x)\, dx = a \cdot \int f(x)\, dx \qquad a \in \mathbb{R}$$

Sums and Differences

$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

Exponential Rule

$$\int f'(x) \cdot e^{f(x)}\, dx = e^{f(x)} + C$$

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Logarithmic Rule

$$\int \frac{f^{'}(x)}{f(x)}\,dx = \ln|f(x)| + C$$

Integration by Parts

$$\int f(x)\cdot g^{'}(x)\,dx = f(x)\cdot g(x) - \int f^{'}(x)g(x)\,dx$$

Integration by Substitution

$$\int f(u(x))\cdot\frac{du}{dx}\,dx = \int f(u)\,du = F(u) + C$$

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3.2 Rules of Integration

Some comments regarding integration by parts:

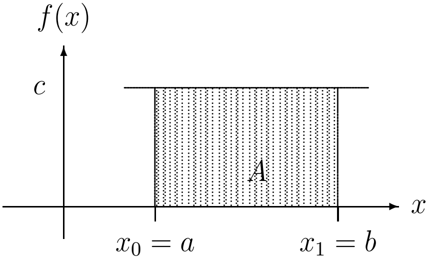
- Integration by parts (formally) requires that a product $f(x) \cdot g'(x)$ is to be integrated. Which factor is to be chosen as $f(x)$ and which one as $g'(x)$ is not determined formally.
  Rule of thumb: Choose the function as $g'(x)$ which is easier integrated.
- Sometimes it is useful to integrate by parts even though a product of the form $f(x) \cdot g'(x)$ is not given in the first place. As second factor one can always use the function 1 which is easily integrated.
  Example: $\int \ln(x)dx$.
- It might sometimes be necessary to apply the integration by parts method more than once. I.e. the integral on the right hand side requires again integration by parts until we have an easy to solve expression.

$\Rightarrow$ There is no specific rule on when and how to apply integration by parts, so only solving lots of exercises will give you a feeling for the way to use it.

3.3 Definite Integrals

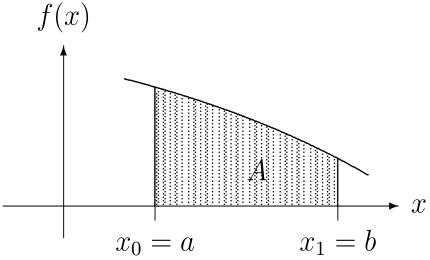
An important application of integration is to calculate the area of many plane regions.

It is easy to calculate the area on the interval $[a; b]$ which lies under the constant function $f(x) = c$: the area is given by: width $\times$ heigth, i.e., $(b - a)\,c$.



3.3 Definite Integrals

For most other functions $f(x)$ which are continuous and nonnegative on the interval $[a; b]$ there is no such formula to determine the area under its graph.



3.3 Definite Integrals

Solution: cut the area in rectangles and let the number of rectangles in the area $a = x_0 < x_1 < \ldots < x_n = b$ go to infinity ($n \to \infty$):

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i \quad \text{with} \quad \Delta x_i = x_i - x_{i-1} \text{ and } x_{i-1} \leq \xi_i \leq x_i \; .$$

If the limits for the inscribed and circumscribed areas exist and are equal, then this limit is called **definite integral** (Rieman Integral):

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

where x is the *variable of integration*; a and b are the *lower* and *upper limit of integration*, respectively.

3.3 Definite Integrals

Definition: *Definite Integral*
Suppose that the function $f(x)$ has an antiderivative $F(x)$ over the intervall $[a; b]$. Then the definite integral is

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Notation: $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = [F(x)]_a^b = \Big|_a^b F(x)$$

3.4 Properties of the Definite Integral

If f is a continuous function in an interval I that contains a, b, and c, then

1. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$
2. $$\int_a^a f(x)\,dx = 0$$
3. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx = \int_c^b f(x)\,dx$$
4. $$\int_a^b [\alpha f(x) + \beta g(x)]\,dx = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx$$

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3.5 Leibniz's Formula

Definition: integral function

Suppose that $f(x)$ is a continuous function over $[a, b]$. Then

$$F_{x_0}(x) = \int_{x_0}^{x} f(t)\, dt$$

is the *integral function* of $f(x)$ for $x, x_0 \in [a, b]$

Differentiation with respect to x (which is a parameter in the integral):

$$\frac{d}{dx} \int_{x_0}^{x} f(t)\, dt = f(x)$$

3.5 Leibniz's Formula

Leibniz's formula

Suppose that $f(x,t)$ and $f'_x(x,t)$ are continuous over the rectangle determined by $a \le x \le b$ and $c \le t \le d$. Suppose that $u(x)$ and $v(x)$ are differentiable functions over $[a,b]$, and that the ranges of u and v are contained in $[c,d]$. Then

$$F(x) = \int_{u(x)}^{v(x)} f(x,t)\,dt$$

and

$$\frac{dF}{dx} = \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x} dt + f(x, v(x)) \cdot \frac{dv(x)}{dx} - f(x, u(x)) \cdot \frac{du(x)}{dx}$$

3.6 Improper Integrals

(a) Infinite intervals of integration

Definition: improper integral (i)

Suppose that $f(x)$ is continuous over $[a; \infty)$ and a is finite. Also, $F(x)$ exists for $f(x)$ on every sub-interval $[a; b]$ with $a < b$. If the limit

$$\lim_{b \to \infty} \int_a^b f(x)\,dx$$

exists and is finite, then we have a converging improper integral of $f(x)$, written as $\int_a^\infty f(x)\,dx$. If the limit does not exist, we have a diverging improper integral.

3.6 Improper Integrals

(b) Diverging integrands

Definition: improper integral (ii)

Suppose that $f(x)$ is continuous over $(a; b]$ and unbounded for $x \to a$, i.e., $\lim_{x \to a} f(x) = \pm\infty$.
If the limit

$$\int_a^b f(x)\,dx = \lim_{c \to a_+} \int_c^b f(x)\,dx$$

exists, we have a converging improper integral of $f(x)$. If the limit does not exist, we have a diverging improper integral.

3.7 Multiple Integrals

Let $f(x_1, \ldots, x_n)$ be a continuous function defined over $[a_1, b_1] \times \cdots \times [a_n, b_n]$. Then

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_n \ldots dx_1$$

is the n-dimensional volume under the surface of f over the area $[a_1, b_1] \times \cdots \times [a_n, b_n]$.

3.7 Multiple Integrals

Let $f(x_1, x_2)$ be a continuous function defined over $[a_1, b_1] \times [a_2, b_2]$. Then

$$\int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x_1, x_2)\, dx_2 \right) dx_1 = \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x_1, x_2)\, dx_1 \right) dx_2$$

(Fubini's theorem)

3.7 Multiple Integrals

Change of variables in double integrals

Suppose that

$$x = g(u, v), \quad y = h(u, v)$$

defines a one-to-one C^1 transformation from an open and bounded set A' in the uv-plane onto an open and bounded set A in the xy-plane, and assume that the Jacobian determinant $\partial(g, h)/\partial(u, v)$ is bounded on A'. Let f be a bounded and continuous function defined on A. Then

$$\int\int_A f(x, y) dx\, dy = \int\int_{A'} f(g(u, v), h(u, v)) \left| \frac{\partial(g, h)}{\partial(u, v)} \right| du\, dv$$