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WIRTSCHAFTS- UND  
SOZIALWISSENSCHAFTLICHE  
FAKULTÄT

Chair of Statistics, Econometrics and Empirical Economics

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**S414**  
**Advanced Mathematical Methods**  
Exercises

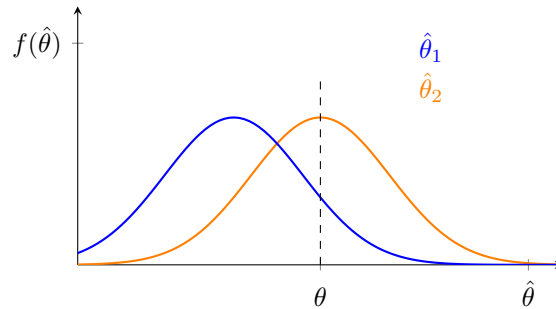
WS 2022/23

**PARAMETER ESTIMATION****EXERCISE 1 Properties of an estimator**

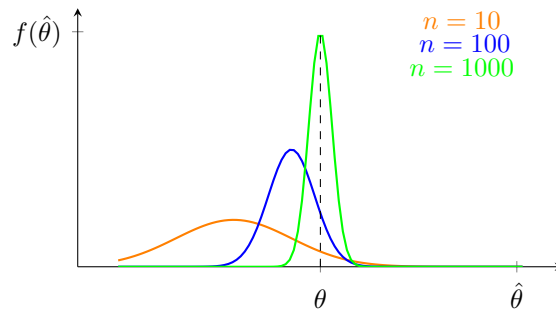
- (a) Illustrate graphically a biased and an unbiased estimator for the parameter  $\theta$ .
- (b) Illustrate graphically a consistent estimator for the parameter  $\theta$ .
- (c) Assume there are two alternative estimators  $\hat{\theta}_1$  and  $\hat{\theta}_2$  for the parameter  $\theta$ . Both estimators are unbiased, however,  $\hat{\theta}_1$  is more efficient than  $\hat{\theta}_2$ . Illustrate this in a graph.
- (d) Decompose the mean squared error (MSE) of an estimator into its components bias and variance.
- (e) Using a graph, explain the possible bias variance trade-off of an estimator (draw two estimators).  
Comprehension question: Why does an estimator have a variance? Why is the estimator a random variable?

**Solution:**

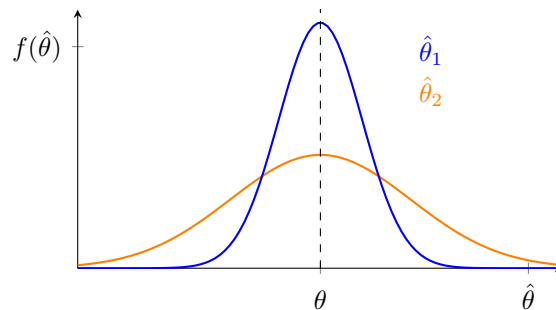
(a)



(b)



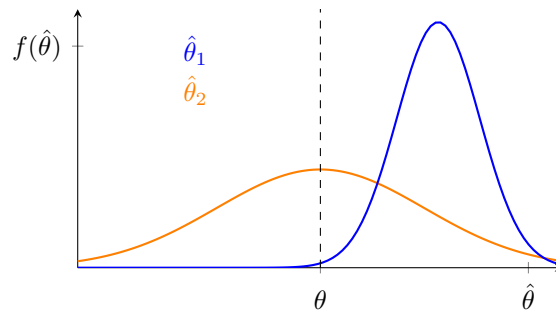
(c)



(d)

$$\begin{aligned}
 MSE &= \mathbb{E}[(\hat{\theta} - \theta)^2] \\
 &= \mathbb{E} \left\{ [(\hat{\theta} - \mathbb{E}(\hat{\theta})) + (\mathbb{E}(\hat{\theta}) - \theta)]^2 \right\} \\
 &= \mathbb{E} \left\{ (\hat{\theta} - \mathbb{E}(\hat{\theta}))^2 + 2(\hat{\theta} - \mathbb{E}(\hat{\theta}))(\mathbb{E}(\hat{\theta}) - \theta) + (\mathbb{E}(\hat{\theta}) - \theta)^2 \right\} \\
 &= \mathbb{E} \left[ (\hat{\theta} - \mathbb{E}(\hat{\theta}))^2 \right] + 2\mathbb{E} \left[ (\hat{\theta} - \mathbb{E}(\hat{\theta}))(\mathbb{E}(\hat{\theta}) - \theta) \right] + \mathbb{E} \left[ (\mathbb{E}(\hat{\theta}) - \theta)^2 \right] \\
 &= \mathbb{E} \left[ (\hat{\theta} - \mathbb{E}(\hat{\theta}))^2 \right] + 2\mathbb{E} \left[ \hat{\theta}\mathbb{E}(\hat{\theta}) - \hat{\theta}\theta - \mathbb{E}(\hat{\theta})^2 + \mathbb{E}(\hat{\theta})\theta \right] + (\mathbb{E}(\hat{\theta}) - \theta)^2 \\
 &= \mathbb{E} \left[ (\hat{\theta} - \mathbb{E}(\hat{\theta}))^2 \right] + 2 \left[ \mathbb{E}(\hat{\theta})^2 - \mathbb{E}(\hat{\theta})\theta - \mathbb{E}(\hat{\theta})^2 + \mathbb{E}(\hat{\theta})\theta \right] + (\mathbb{E}(\hat{\theta}) - \theta)^2 \\
 &= \mathbb{E} \left[ (\hat{\theta} - \mathbb{E}(\hat{\theta}))^2 \right] + \left[ \mathbb{E}(\hat{\theta}) - \theta \right]^2 \\
 &= \text{Var}(\hat{\theta}) + \text{Bias}(\hat{\theta})^2
 \end{aligned}$$

(e)



Estimators are measurable functions of random variables and thus, random variables themselves.

**EXERCISE 2 Method of Moments and Maximum Likelihood**

- (a) Assume that in the population the random variable  $X$  follows a Poisson distribution with parameter  $\lambda$ . Propose a method of moment (MM) estimator for the parameter  $\lambda$ . What do you need for the MM estimator? Are there any further possible estimators?
- (b) Assume that in the population the random variable  $X$  follows an Exponential distribution with parameter  $\lambda$ . Propose a MM estimator for the parameter  $\lambda$ . Justify your choice.
- (c) Assumptions as in (a): Derive the maximum likelihood (ML) estimator for the parameter  $\lambda$  of the Poisson distribution.
- (d) Assumptions as in (b): Derive the ML estimator for the parameter  $\lambda$  of the Exponential distribution.

**Solution:**

(a)  $X \sim Po(\lambda), \mathbb{E}[X] = \lambda, Var[X] = \lambda$

Idea: Set theoretical moments equal to empirical moments to obtain the MM estimator

$$\hat{\lambda}_1 = \frac{1}{n} \sum_{i=1}^n x_i$$

$$\hat{\lambda}_2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - \left( \frac{1}{n} \sum_{i=1}^n x_i \right)^2$$

(b)  $X \sim Ex(\lambda)$ , theoretical moments:  $\mathbb{E}[X] = \frac{1}{\lambda}, Var[X] = \frac{1}{\lambda^2}$

Solve for  $\lambda$ :

$$\lambda = \frac{1}{\mathbb{E}(X)}$$

$$\lambda = \frac{1}{\sqrt{Var(X)}} = \frac{1}{\sqrt{\mathbb{E}(X^2) - \mathbb{E}(X)^2}}$$

Replace theoretical by empirical moments:

$$\hat{\lambda}_1 = \frac{1}{\frac{1}{n} \sum_{i=1}^n x_i}$$

$$\hat{\lambda}_2 = \frac{1}{\sqrt{\frac{1}{n} \sum_{i=1}^n x_i^2 - \left( \frac{1}{n} \sum_{i=1}^n x_i \right)^2}}$$

(c)

$$\mathcal{L}(\tilde{\lambda}) = \mathcal{L}(x_1, \dots, x_n; \tilde{\lambda}) = \prod_{i=1}^n \frac{\tilde{\lambda}^{x_i}}{x_i!} e^{-\tilde{\lambda}}$$

$$\ln(\mathcal{L}(x_1, \dots, x_n; \tilde{\lambda})) = \ln \left[ \prod_{i=1}^n \frac{\tilde{\lambda}^{x_i}}{x_i!} e^{-\tilde{\lambda}} \right]$$

$$= \sum_{i=1}^n \ln \left[ \frac{\tilde{\lambda}^{x_i}}{x_i!} e^{-\tilde{\lambda}} \right]$$

$$= \sum_{i=1}^n \left[ \ln(\tilde{\lambda}^{x_i}) - \ln(x_i!) - \tilde{\lambda} \right]$$

$$= \sum_{i=1}^n \left[ x_i \cdot \ln(\tilde{\lambda}) - \ln(x_i!) - \tilde{\lambda} \right]$$

FOC :

$$\frac{\partial \ln \mathcal{L}(\tilde{\lambda})}{\partial \tilde{\lambda}} = \sum_{i=1}^n \left[ x_i \frac{1}{\tilde{\lambda}} - 1 \right] \stackrel{!}{=} 0$$

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^n x_i$$

(d)

$$\begin{aligned}\mathcal{L}(x_1, \dots, x_n; \tilde{\lambda}) &= \prod_{i=1}^n \tilde{\lambda} e^{-\tilde{\lambda} x_i} \\ \ln(\mathcal{L}(x_1, \dots, x_n; \tilde{\lambda})) &= \sum_{i=1}^n \left[ \ln(\tilde{\lambda} e^{-\tilde{\lambda} x_i}) \right] \\ &= n \cdot \ln(\tilde{\lambda}) - \tilde{\lambda} \sum_{i=1}^n x_i\end{aligned}$$

*FOC :*

$$\begin{aligned}\frac{\partial \ln \mathcal{L}(\tilde{\lambda})}{\partial \tilde{\lambda}} &= \frac{n}{\tilde{\lambda}} - \sum_{i=1}^n x_i \stackrel{!}{=} 0 \\ \hat{\lambda} &= \frac{1}{\frac{1}{n} \sum_{i=1}^n x_i}\end{aligned}$$