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# S414
# Advanced Mathematical Methods

Exercises

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# LINEAR ALGEBRA

## EXERCISE 1 Vector product

Calculate for $\mathbf{v}' = (-1 \quad 0 \quad 3 \quad -2)$ and $\mathbf{w} = (2 \quad 3 \quad -1 \quad -3)'$ the following expressions:

a) $\mathbf{v}'\mathbf{w}$ b) $\mathbf{v}'\mathbf{v}$

c) $\mathbf{w}' \cdot \mathbf{w} \cdot \mathbf{v}$ d) $\mathbf{w} \cdot \mathbf{v}' \cdot \mathbf{w}$

In subtasks c) and d) consider the dimensions of the vectors in order to figure out which product has to be calculated first.

## EXERCISE 2 Orthogonality

Determine the components $x$, $y$ and $z$ in a way, such that the vectors $\boldsymbol{v}_1 = (1 \quad 2 \quad -1)'$, $\boldsymbol{v}_2 = (4 \quad 2 \quad x)'$ and $\boldsymbol{v}_3 = (y \quad z \quad 1)'$ are pairwise orthogonal to each other.

## EXERCISE 3 Linear Combination

The vectors $\mathbf{v_1} = (1 \quad 1 \quad 1)'$, $\mathbf{v_2} = (1 \quad 2 \quad 3)'$ and $\mathbf{v_3} = (2 \quad -1 \quad 1)'$ are given. Show that the vector $\mathbf{w} = (1 \quad -2 \quad 5)'$ can be described as a linear combination of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$.

## Exercise 4 Matrix Multiplication

Decide whether the matrix multiplications $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$ are possible and if so, carry them out.

a) $\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \end{pmatrix}$ $\quad \mathbf{B} = \begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 4 \end{pmatrix}$

b) $\mathbf{A} = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 2 & -3 \end{pmatrix}$ $\quad \mathbf{B} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$

c) $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $\quad \mathbf{B} = \begin{pmatrix} -2 & 0 & 1 \\ 4 & -1 & 2 \end{pmatrix}$

d) $\mathbf{A} = \begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}$ $\quad \mathbf{B} = \begin{pmatrix} 3 & 5 & 2 \\ -4 & -9 & 2 \\ -2 & -8 & 1 \end{pmatrix}$

e) $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ $\quad \mathbf{B} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

f) $\mathbf{A} = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 6 & 6 \\ 3 & 7 & 4 \\ 0 & 8 & 2 \end{pmatrix}$ $\quad \mathbf{B} = \mathbf{A}'$

## Exercise 5 Matrix Algebra

Apply the calculation rules for matrix algebra in order to simplify the following expressions:

a) $\mathbf{A}(\mathbf{BA})^{-1}\mathbf{B}$ b) $(\mathbf{AB}')'(\mathbf{BA}')^{-1}\mathbf{C}$ c) $\mathbf{AB}'(\mathbf{B}^{-1})'\mathbf{A}^{-1}$

## Exercise 6 Matrix Algebra

Solve each of the following matrix equations for $\mathbf{X}$ applying the calculation rules for matrices:

a) $\mathbf{A}'\mathbf{I} + \mathbf{X}' = [\mathbf{A}(\mathbf{I} + \mathbf{B})]'$
b) $(\mathbf{XA} + \mathbf{IX})' = \mathbf{A}' + \mathbf{I}$
c) $\mathbf{X}(\mathbf{A} + \mathbf{I}) = \mathbf{I} + \mathbf{A}^{-1}$

$\mathbf{I}$: appropriate identity matrix

## EXERCISE 7 **Determinant**

Calculate the determinant for the following matrices:

a) $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ -4 & 2 \end{pmatrix}$

b) $\mathbf{B} = \begin{pmatrix} 1 & -4 & -10 \\ 10 & -8 & 2 \\ 0 & -1 & 6 \end{pmatrix}$

c) $\mathbf{C} = \begin{pmatrix} -7 & 1 & -10 \\ 1 & 10 & 2 \\ 1 & 0 & 6 \end{pmatrix}$

d) $\mathbf{D} = \begin{pmatrix} -3 & 0 & -8 & 7 \\ -7 & 1 & -4 & -10 \\ 1 & 10 & -8 & 2 \\ 1 & 0 & -1 & 6 \end{pmatrix}$

e) $\mathbf{E} = \mathbf{D}^{-1}$

f) $\mathbf{F} = \begin{pmatrix} 3 & 0 & 8 & -7 \\ 7 & -1 & 4 & 10 \\ -1 & -10 & 8 & -2 \\ -1 & 0 & 1 & -6 \end{pmatrix}$

## EXERCISE 8 **Determinant**

The matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 5 & 7 \\ 3 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 5 \\ 1 & 0 & 1 \end{pmatrix}$$

are given.

a) Compute the determinant of $\mathbf{A}$.

b) Compute the determinant of a matrix that we receive by interchanging the first and the third column of $\mathbf{A}$ (second column $\mathbf{A}$ unchanged). Compare your result to a).

c) Compute the determinant of $\mathbf{A}'$. Compare your result to a).

d) Compute the determinant of $2 \cdot \mathbf{A}$. How can you compute this determinant more quickly?

e) Compute the determinant of $\mathbf{B}$.

f) Compute the determinant of $\mathbf{A}\,\mathbf{B}$. Compare your result to a) and e).

g) Compute the determinant of $\mathbf{A} + \mathbf{B}$. Compare your result to $det(\mathbf{A}) + det(\mathbf{B})$.

## Exercise 9 Calculation of the Inverse

Calculate the inverse if possible:

a) $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ -4 & 2 \end{pmatrix}$  b) $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ -2 & -0,5 \end{pmatrix}$

c) $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$  d) $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -1 & 3 \end{pmatrix}$

## Exercise 10 Rank, Regularity and Inverse

The following matrix is given:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{pmatrix}$$

a) Determine the rank of the matrix.

b) Is the matrix **A** regular or singular?

c) Calculate the inverse of **A**, if possible.

## Exercise 11 Linear Equation Systems

Determine for the following linear equation system the solution vector using

a) Gaussian Elimination

b) Matrix Inversion

c) Cramer's Rule

$$\begin{aligned} x_2 + x_3 &= -1 \\ 3x_1 + 4x_2 + 5x_3 &= -2 \\ 4x_1 + 6x_2 + 8x_3 &= -4 \end{aligned}$$

## Exercise 12 Rank of a matrix

Determine the rank of the following matrices using Gaussian Elimination:

a) $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 & 1 & 2 \\ -1 & 2 & 1 & 0 & -1 \\ 0 & 4 & 1 & 2 & 1 \end{pmatrix}$  b) $\mathbf{B} = \begin{pmatrix} 4 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & -1 & -1 \end{pmatrix}$

## Exercise 13 **Inverse of a matrix**

Given the follwing matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}$$

Calculate the inverse $A^{-1}$ using Gaussian Elimination.

**Solution Exercise 1:**

a) 1

b) 14

c) $\begin{pmatrix} -23 \\ 0 \\ 69 \\ -46 \end{pmatrix}$

d) $\begin{pmatrix} 2 \\ 3 \\ -1 \\ -3 \end{pmatrix}$

**Solution Exercise 2:**
x=8, z=2, y=-3

**Solution Exercise 3:**
$-6 \cdot \mathbf{v}_1 + 3 \cdot \mathbf{v}_2 + 2 \cdot \mathbf{v}_3 = \mathbf{w}$

**Solution Exercise 4:**

a) Possible. $\mathbf{C} = \begin{pmatrix} -1 & 1 & 4 & 3 \\ 4 & 8 & -2 & 12 \end{pmatrix}$

b) Not possible. $\underset{(2\times2)}{\mathbf{B}} \cdot \underset{(2\times3)}{\mathbf{A}} = \begin{pmatrix} -4 & 7 & 0 \\ 0 & 5 & -4 \end{pmatrix}$

c) Not possible. $\underset{(2\times3)}{\mathbf{B}} \cdot \underset{(3\times1)}{\mathbf{A}} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

d) Possible, both sides. $7 \cdot \mathbf{I}$

e) Possible, both sides. $\mathbf{I}$

f) Possible, both sides.
$\begin{pmatrix} 26 & 32 & 38 & 40 \\ 32 & 76 & 72 & 60 \\ 38 & 72 & 74 & 64 \\ 40 & 60 & 64 & 68 \end{pmatrix}$
and $\begin{pmatrix} 14 & 38 & 24 \\ 38 & 174 & 80 \\ 24 & 80 & 56 \end{pmatrix}$

**Solution Exercise 5:**
a) $\mathbf{I}$
b) $\mathbf{C}$
c) $\mathbf{I}$

## Solution Exercise 6:

a) $\mathbf{X} = \mathbf{AB}$

b) $\mathbf{X} = \mathbf{I}$

c) $\mathbf{X} = \mathbf{A^{-1}}$

## Solution Exercise 7:

a) $|\mathbf{A}| = 12$

b) $|\mathbf{B}| = 294$

c) $|\mathbf{C}| = -324$

d) $|\mathbf{D}| = 989$

e) $|\mathbf{E}| = \mathbf{\frac{1}{989}}$

f) $|\mathbf{F}| = 989$

## Solution Exercise 8:

a) $|\mathbf{A}| = -26$

b) $\begin{vmatrix} 3 & 1 & 0 \\ 7 & 5 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 26$

c) $|\mathbf{A}'| = -26$

d) $|2\mathbf{A}| = -208$

e) $|\mathbf{B}| = 17$

f) $|\mathbf{AB}| = -442$

g) $|\mathbf{A} + \mathbf{B}| = 92$

## Solution Exercise 9:

a) $\mathbf{A^{-1}} = \frac{1}{12} \cdot \begin{pmatrix} 2 & -1 \\ 4 & 4 \end{pmatrix}$

b) $\mathbf{B}$ is singular $\Leftrightarrow$ $\mathbf{B^{-1}}$ doesn't exist!

c) $\mathbf{C^{-1}} = -\frac{1}{10} \begin{pmatrix} -4 & -2 \\ -3 & 1 \end{pmatrix}$

d) $\mathbf{D^{-1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

## Solution Exercise 10:

a) $Rg(\mathbf{A}) = 3.$

b) $\mathbf{A}$ is regular.

c) $\mathbf{A}^{-1} = \begin{pmatrix} 0 & -0,2 & 0,3 \\ -1 & 0,6 & 0,1 \\ 0 & 0,4 & -0,1 \end{pmatrix}.$

## Solution Exercise 11:

Solution vector: $x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

## Solution Exercise 12:

a) The rank of the matrix is 3.
b) The rank of the matrix is 3.

## Solution Exercise 13:

$$\mathbf{A^{-1}} = \begin{pmatrix} 6,75 & -2,75 & 0,75 \\ -2,75 & 1,25 & -0,25 \\ 0,75 & -0,25 & 0,25 \end{pmatrix}$$