Advanced Mathematical Methods

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6 Statistical Inference

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Readings

- A. Papoulis and A. U. Pillai. *Probability, Random Variables and Stochastic Processes.*
  Mc Graw Hill, fourth edition, 2002, Chapter 8

Online References

MIT Course on Probabilistic Systems Analysis and Applied Probability (by John Tsitsiklis)

- Lecture 25: Classical Inference III

Hypothesis testing

Ingredients:

- null hypothesis H_0, alternative hypothesis H_1
- significance level α (given)

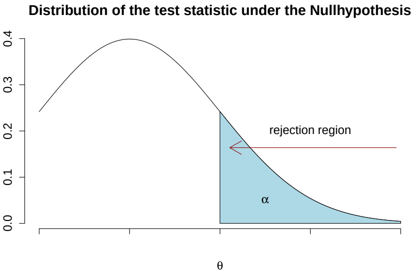
2 possible errors:

- α - error/ type 1 error:
  reject a correct (null) hypothesis
- β - error/ type 2 error:
  do not reject a wrong (null) hypothesis

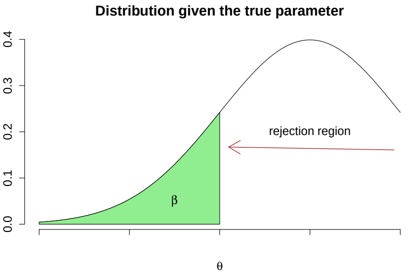
Two ways of testing

θ unknown parameter in the population

1. $H_0 : \theta = \theta_0$
   $H_1 : \theta \neq \theta_0$
   $\rightarrow$ two-sided test
2. $H_0 : \theta \leq \theta_0$ $\quad$ $H_0 : \theta \geq \theta_0$
   $H_1 : \theta > \theta_0$ $\quad$ $H_1 : \theta < \theta_0$
   $\rightarrow$ one-sided test



- $f_q(q, \theta_0)$: distribution under the H_0



- $f_q(q, \theta)$: distribution given the true θ

- Under H_1, the most likely values of q are on the right of $f_q(q, \theta_0)$.
- We therefore reject H_0 if $q > c$ (with rejection area $[c, \infty]$)
- We select α: $P(q > c|H_0) = \alpha \quad \rightarrow \quad c = q_{1-\alpha}$
  and don't reject H_0 if $q < q_{1-\alpha}$

Operating characteristic:

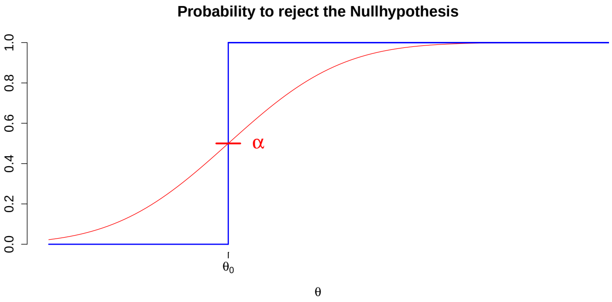
$$\underbrace{\beta(\theta)}_{\text{depends on } \theta \text{, the true parameter}} = \int_{-\infty}^{c} f_q(q, \theta) dq$$

$\rightarrow$ can't be controlled

Ideal Situation:

$\alpha = \beta = 0$

for H_0: $\theta = \theta_0$ and H_1: $\theta > \theta_0$



Ideally:

- don't reject H_0 as long as the true value θ is smaller than θ_0
- reject as soon as θ is greater than θ_0

α: **at the intersection:**

if α is small, the chances to reject H_0 are small if θ is only slightly bigger than θ_0

The faster the probability to reject H_0 increases (steeper red line), the better.

Hence: **power of the test**

What does significant really mean?

statistical significance

- does not answer the question wether the null hypothesis is wrong or right
- does not indicate how (un-) likely the null hypothesis is
- only controlled by maximum probability to run into type 1 error (α)
- provides no control over probability of type 2 error (β)

goal: for α given

$\rightarrow$ minimal β

$\rightarrow$ minimal $\alpha + \beta$

$\rightarrow$ maximal $1 - \beta$

t-Test

estimated parameters $\widehat{\beta_1} \ldots \widehat{\beta_k}$

1. define H_0, e.g. $H_0 : \beta_k = \bar{\beta}_k$
2. define H_1, e.g. $H_1 : \beta_k \neq \bar{\beta}_k$
3. believe in law of large numbers and CLT
4. construct test statistic

$$t = \frac{\widehat{\beta_k} - \bar{\beta}_k}{s.e.(\widehat{\beta_k})} \sim t(N-K) \quad \text{under} \quad H_0$$

5. choose significance level α
6. compare t and critical value
   compare t and empirical p-value

Confidence Interval

construct a confidence interval around $\widehat{\beta_k}$
$\rightarrow$ interval for $\bar{\beta}_k$, for which $H_0 : \beta_k = \bar{\beta}_k$ cannot be rejected

$$CI(\beta_k, \alpha) = \left[\widehat{\beta_k} - t_{\frac{\alpha}{2}} \cdot s.e.(\widehat{\beta}), \widehat{\beta_k} + t_{\frac{\alpha}{2}} \cdot s.e.(\widehat{\beta})\right]$$

Testing linear hypotheses: Wald test

Multiple Hypotheses ($\#r$) for multiple parameters (k)

$$\underbrace{\boldsymbol{R}}_{\#r\times k}\underbrace{\boldsymbol{\beta}}_{k\times 1} = \underbrace{\boldsymbol{r}}_{\#r\times 1}$$

under H_0:

$$\boldsymbol{R}\widehat{\boldsymbol{\beta}} \underset{p}{\rightarrow} \boldsymbol{r} \qquad \boldsymbol{R}\widehat{\boldsymbol{\beta}} \overset{a}{\sim} N(0, \boldsymbol{R}Var(\widehat{\boldsymbol{\beta}})\boldsymbol{R}')$$

$$\underbrace{(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - \boldsymbol{r})'(\boldsymbol{R}Var(\widehat{\boldsymbol{\beta}})\boldsymbol{R}')^{-1}(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - \boldsymbol{r}) \overset{a}{\sim} \chi^2(\#r)}_{\text{Wald test statistic for linear hypotheses}}$$