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WIRTSCHAFTS- UND  
SOZIALWISSENSCHAFTLICHE  
FAKULTÄT

Chair of Statistics, Econometrics and Empirical Economics

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**S414**  
**Advanced Mathematical Methods**  
Exercises

WS 2020/21

**PROBABILITY AND DISTRIBUTION THEORY****EXERCISE 1 Probability and Distribution Theory**

Given a continuous random variable  $X$  with:

$$f(x) = \begin{cases} 4ax & 0 \leq x < 1 \\ -ax + 0.5 & 1 \leq x \leq 5 \\ 0 & \text{else} \end{cases}$$

Determine the parameter  $a$  such that  $f(x)$  is a density function of  $X$ . Calculate the corresponding distribution function and sketch it. Compute the expectation and the variance of  $X$ .

**EXERCISE 2 Probability and Distribution Theory**

The Federal Statistical Office assumes all values in the interval  $2 \leq x \leq 3$  to be possible realizations of the random variable  $X$  : "Growth rate of the GDP". Moreover, the following function is assumed:

$$f(x) = \begin{cases} c \cdot (x - 2) & 2 \leq x \leq 3 \\ 0 & \text{else} \end{cases}$$

- Determine  $c$  such that the function  $f(x)$  is a density function of the random variable  $X$ .
- Compute the distribution function of the random variable  $X$ .
- Compute  $P(X < 2.1)$  and  $P(2.1 < X < 2.8)$ .
- Compute  $P(-4 \leq X \leq 3 | X \leq 2.1)$  and show that the events  $\{-4 \leq X \leq 3\}$  and  $\{X \leq 2.1\}$  are statistically independent.
- Compute the expectation, median and the variance of  $X$ .

**EXERCISE 3 Probability and Distribution Theory**

Show the Markov - inequality:

$$P(X \geq c) \leq \frac{E[X]}{c}$$

for every positive value of  $c$  with  $X$  being strictly non-negative.

**EXERCISE 4 Probability and Distribution Theory**

		$X$		
		1	2	3
$Y$	1	0.25	0.15	0.10
	2	0.10	0.15	0.25

- a) Compute the expectation and the variance of  $X$  and  $Y$ .
- b) Determine the conditional distributions of  $X|Y = y$  and  $Y|X = x$ .
- c) Determine the covariance and the correlation coefficient of  $X$  and  $Y$ .
- d) Determine the variance of  $X + Y$ .

**EXERCISE 5 Probability and Distribution Theory**

The joint probability function of  $X$  and  $Y$  is given by:

$$f(x, y) = \begin{cases} e^{-2\lambda} \cdot \frac{\lambda^{x+y}}{x!y!} & x, y \in \{0, 1, \dots\} \\ 0 & \text{else} \end{cases}$$

- a) Determine the marginal distributions of  $X$  and  $Y$ .
- b) Determine the conditional distributions of  $X|Y = y$  and  $Y|X = x$  and compare them to the marginal distributions.
- c) Determine the covariance of  $X$  and  $Y$ .

Hint:  $e^\lambda = \sum_y \frac{\lambda^y}{y!}$

**EXERCISE 6 Probability and Distribution Theory**

Suppose that  $x_u$  is the  $u$  percentile of the random variable  $X$ , that is,  $F(x_u) = u$ . Show that if  $f(-x) = f(x)$ , then  $x_{1-u} = -x_u$

**EXERCISE 7 Probability and Distribution Theory**

If  $X \sim N(1000, 400)$  find:

- a)  $P(X < 1024)$
- b)  $P(X < 1024|X > 961)$
- c)  $P(31 < \sqrt{X} < 32)$

**EXERCISE 8 Probability and Distribution Theory**

A fair coin is tossed three times and the random variable  $X$  equals the total number of heads. Find and sketch  $F_X(x)$  and  $f_X(x)$ .

**EXERCISE 9 Probability and Distribution Theory**

The random variable  $X$  is  $N(5, 2)$  and  $Y = 2X + 4$ . Find  $\mu_Y$ ,  $\sigma_Y$  and  $f_Y(y)$ .

**EXERCISE 10 Probability and Distribution Theory**

Find  $F_Y(y)$  and  $f_Y(y)$  if  $Y = -4X + 3$  and  $f_X(x) = 2e^{-2x}$  with  $x \geq 0$ .

**EXERCISE 11 Probability and Distribution Theory**

The random variable  $X$  is uniform in the interval  $[-2c, 2c]$ . Find and sketch  $f_Y(y)$  and  $F_Y(y)$  if  $Y = g(X)$  and  $g(X) = x^2$ .

**EXERCISE 12 Probability and Distribution Theory**

If  $X$  is  $N(0, 4)$  and  $Y = 3x^2$ , find  $\mu_Y$  and  $\sigma_Y^2$ .

**EXERCISE 13 Probability and Distribution Theory**

The random variables  $X$  and  $Y$  are  $N(\mu_x, \sigma_x^2, \mu_y, \sigma_y^2, \rho_{xy}) = N(3, 4, 1, 4, 0.5)$ . Find  $f(y|x)$  and  $f(x|y)$ .

**Solution Exercise 1:**

$$a = \frac{1}{10}$$

$$\mathbb{E}[x] = 2$$

$$\text{var}(x) = 1.166\bar{6}$$

**Solution Exercise 2:**

a)  $c = 2$

b)

$$F(x) = \begin{cases} 0 & \text{for } x < 2 \\ x^2 - 4x + 4 & \text{for } 2 \leq x \leq 3 \\ 1 & \text{for } x > 3 \end{cases}$$

c)  $P(X < 2.1) = \underline{0.01}$   
 $P(2.1 < X < 2.8) = \underline{0.63}$

d)

$$P(-4 \leq X \leq 3 | X \leq) = \underline{1}$$

The events  $A = \{-4 \leq X \leq 3\}$  and  $B = \{x \leq 2.1\}$  are independent if  $P(B|A) = P(B)$ .

We have:  $P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$

with  $P(A) = P(-4 \leq X \leq 3) = 1$ ,

and  $P(A|B) = P(-4 \leq X \leq 3 | x \leq 2.1) = 1$ .

Hence:  $P(A \cap B) = P(B|A) = P(B)$  q.e.d.

e)

- $\underline{\underline{\bar{x}[0.5] = 2 + \frac{1}{\sqrt{2}} = 2.7071}}$
- $\underline{\underline{\mathbb{E}[x] = 2.66\bar{6}}}$
- $\underline{\underline{\text{var}[x] = \frac{1}{18} = 0.05\bar{5}}}$

**Solution Exercise 3:**

$$\mathbb{E}[x] = \int_{-\infty}^c xf(x)dx + \int_c^{\infty} xf(x)dx$$

$$\mathbb{E}[x] > \int_c^{\infty} xf(x)dx$$

$$\begin{aligned} \mathbb{E}[x] &> c \int_c^{\infty} f(x)dx \\ &> cP(X \geq c) \end{aligned}$$

$$\frac{\mathbb{E}[x]}{c} = P(X \geq c) \text{ q.e.d.}$$

**Solution Exercise 4:**

a)  $\mathbb{E}[x] = 2$   
 $\mathbb{E}[x] = 1.5$

b) The conditional probability distributions:

		$x = 1$	$x = 2$	$x = 3$
Conditional	$y = 1$	0.5	0.3	0.2
distribution $f(x y)$	$y = 2$	0.2	0.3	0.5
		$x = 1$	$x = 2$	$x = 3$
Conditional	$y = 1$	5/7	1/2	2/7
distribution $f(x y)$	$y = 2$	2/7	1/2	5/7

c)  $\text{cov}[x, y] = \underline{0.15}$

d)  $\text{var}[x + y] = 1.25$

**Solution Exercise 5:**

a)  $f(x) = e^{-\lambda} \frac{\lambda^x}{x!}$  and  $f(y) = e^{-\lambda} \frac{\lambda^y}{y!}$

b)  $f(x|y) = f(x)$

$$f(y|x) = f(y)$$

c)  $\text{cov}[x, y] = 0$

**Solution Exercise 6:**

If  $f(x) = f(-x)$  then  $\int_{-\infty}^{-x_u} f(z)dz = \int_{x_u}^{\infty} f(z)dz$ .

From which follows that:

$$F(-x_u) = 1 - F(x_u) = 1 - u$$

Hence,  $-x_u = x_{1-u}$  q.e.d.

**Solution Exercise 7:**

a)  $P(X < 1024) = 0.8849$

b)  $P(X < 1024 | X > 961) = 0.8819$

c)  $P(31 < \sqrt{x} < 32) = 0.8593$

**Solution Exercise 8:**

$$f_X(x) = 0.5^3 \binom{3}{x} = 0.5^3 \frac{3!}{x!(3-x)!}$$

$$F_X(x) = 0.5^3 \sum_{k=1}^x \binom{3}{k} = 0.5^3 \sum_{k=1}^x \frac{3!}{k!(3-k)!}$$

**Solution Exercise 9:**

$$\mu_y = 14$$

$$\sigma_y = 2\sqrt{2}$$

$$f(y) = \frac{1}{4\sqrt{\pi}} e^{-\frac{(y-14)^2}{16}}$$

**Solution Exercise 10:**

$$f_Y(y) = 0.5e^{\frac{y-3}{2}}$$

$$F_Y(y) = e^{\frac{y-3}{2}}$$

**Solution Exercise 11:**

$$f_Y(y) = \frac{1}{4c} \frac{1}{\sqrt{y}} \text{ for } y \in [0; 4c^2]$$

$$F_Y(y) = \frac{1}{2c} \sqrt{y} \text{ for } y \in [0; 4c^2]$$

**Solution Exercise 12:**

$$\mu_Y = \mathbb{E}[Y] = 12$$

$$\sigma_Y^2 = \text{var}[Y] = 288$$

**Solution Exercise 13:**

$$Y|X \sim N\left(\mu_y + \rho_{xy} \frac{\sigma_y}{\sigma_x}(x - \mu_x); \sigma_y^2(1 - \rho_{xy}^2)\right)$$

$$X|Y \sim N\left(\mu_x + \rho_{xy} \frac{\sigma_x}{\sigma_y}(y - \mu_y); \sigma_x^2(1 - \rho_{xy}^2)\right)$$