4th set of assignment Financial Econometrics

1. GMM inference

For the GMM estimator $\hat{b}_{GMM}$ resulting from

$$\underset{\{\hat{b}\}}{argmin}\; g_T(\hat{b})'W g_T(\hat{b}) \tag{1}$$

We have $\hat{b}_{GMM} \underset{p}{\longrightarrow} b$ and
$\sqrt{T}(\hat{b}_{GMM} - b) \underset{d}{\longrightarrow} N(0, Avar(\hat{b}_{GMM}))$

Where $Avar(\hat{b}_{GMM})$ denotes the asymptotic variance covariance matrix.
In a finite sample we use the approximation

$$\hat{b}_{GMM} \overset{a}{\sim} N(b, \frac{Avar(\hat{b}_{GMM})}{T}) \tag{2}$$

to test hypotheses about b.

We have $Avar(\hat{b}_{GMM}) = (d'wd)^{-1}d'wSwd(d'wd)^{-1}$.

To compute $Avar(\hat{b}_{GMM})$ you need to write

$$d = \frac{\partial g_T(b)}{\partial b'} \quad . \tag{3}$$

$g_T(b)$ is a vector valued function, i.e. it returns, for a given parameter vector $b = (b_1, b_2, ..., b_k)'$, the vector of sample moments:

$$\begin{pmatrix} E_T(u_t^1(b)) \\ \vdots \\ E_T(u_t^N(b)) \end{pmatrix} = \begin{pmatrix} \frac{1}{T}\sum_{t=1}^T u_t^1(b) \\ \vdots \\ \frac{1}{T}\sum_{t=1}^T u_t^N(b) \end{pmatrix} \tag{4}$$

$d = \frac{\partial g_T(b)}{\partial b'}$ is then

$$\begin{pmatrix} \frac{\partial E_t(u_t^1(b))}{\partial b_1} & \dots & \frac{\partial E_t(u_t^1(b))}{\partial b_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial E_t(u_t^N(b))}{\partial b_1} & \dots & \frac{\partial E_t(u_t^N(b))}{\partial b_k} \end{pmatrix} N \times K \tag{5}$$

Write d <u>in detail</u> for the GMM estimation framework of the consumption based model where

$$m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} . \tag{6}$$

Use two moment restrictions for two asset returns $R^a_{t+1}$and R^b_{t+1}:

$$E(m_{t+1}R^a_{t+1} - 1) = 0 \tag{7}$$

$$E(m_{t+1}R^b_{t+1} - 1) = 0 \tag{8}$$

What is b?
What is $u_t(b)$?
What is $E_t(u_t(b))$ and $g_T(b)$?
What is $\frac{\partial g_T(b)}{\partial b'}$?

Write all in greatest detail!

You have succeeded in computing a consistent estimate of $Avar(\hat{b}_{GMM})$ for your GMM application.

$$\widehat{Avar(\hat{b}_{GMM})} = \begin{pmatrix} 5 & 0.3 \\ 0.3 & 10 \end{pmatrix} \tag{9}$$

You have used $T = 100$ observations. Your GMM estimates are given by

$$\hat{\beta}_{GMM} = 0.8 \quad \hat{\gamma}_{GMM} = 0.1 \tag{10}$$

Compute an estimate of $Var(\hat{\beta}_{GMM})$ and $Var(\hat{\gamma}_{GMM})$.
Test the hypotheses

$$\begin{array}{cc} H_0 : \quad \beta = 1 & H_0 : \quad \gamma = 0 \\ \text{versus} & \text{versus} \\ H_A : \quad \beta \neq 1 & H_A : \quad \gamma \neq 0 \end{array} \tag{11}$$

using the t-statistics

$$t_1 : \frac{\hat{\beta}_{GMM} - 1}{\sqrt{\widehat{Var}(\hat{\beta}_{GMM})}} \qquad t_2 : \frac{\hat{\gamma}_{GMM}}{\sqrt{\widehat{Var}(\hat{\gamma}_{GMM})}} \tag{12}$$

t_1 and t_2 are approximately N(0,1) under the respective Null-Hypothesis.

2. Application of the δ-method

Suppose you have obtained a GMM estimator for $b = \begin{bmatrix} \theta \\ \phi \end{bmatrix}$ *i.e.* $\hat{b} = \begin{bmatrix} \hat{\theta} \\ \hat{\phi} \end{bmatrix}$.

We have

$$\sqrt{T}(\hat{b} - b) \underset{d}{\longrightarrow} N(0, \Sigma) \tag{13}$$

where Σ is the asymptotic variance covariance matrix.

A consistent estimate of Σ, denoted $\widehat{\Sigma}$, is given by

$$\widehat{\Sigma} = \begin{pmatrix} 2 & 0.2 \\ 0.2 & 3 \end{pmatrix} \tag{14}$$

The sample has $T = 100$ observations.
Provide estimates of $Var(\hat{\theta})$ and $Var(\hat{\phi})$ using this information. The GMM estimates are $\hat{\theta} = 0.6$ and $\hat{\phi} = 0.4$

You are interested in testing whether

$$r = \frac{\phi}{\phi + \theta} = 0.5 \tag{15}$$

Construct a suitable test statistic (again, a t-statistic). For this purpose compute an estimate of the variance of $\hat{r} = \frac{\hat{\phi}}{\hat{\phi}+\hat{\theta}}$, $Var(\hat{r})$, by using the δ-method.

Hints:

$$a(b) = \frac{\phi}{\phi + \theta} = r \tag{16}$$

$$\hat{r} = a(\hat{b}) \underset{p}{\longrightarrow} a(b) \tag{17}$$

$$\sqrt{T}(a(\hat{b}) - a(b)) \underset{d}{\longrightarrow} N(0, A(b)\Sigma A(b)') \tag{18}$$

where $A(b) = \frac{\partial a(b)}{\partial b'} = \left(\frac{\partial a(b)}{\partial \phi}, \frac{\partial a(b)}{\partial \theta} \right)$

The test statistic is

$$t = \frac{\hat{r} - 0.5}{\sqrt{\widehat{Var}(\hat{r})}} \tag{19}$$

t is approximately N(0,1) under the Null Hypothesis that $r = 0.5$.