# **Advanced Econometrics -** $1^{\textbf{st}}$ assignment sheet

*Task 1*

Show that these functions are Kernels:

$$K(\psi) = (2\pi)^{-1/2}\exp(-0.5\psi^2) \tag{1}$$
$$K(\psi) = \mathbf{1}\{|\psi| \leq 1\}3/4 \cdot \{1-\psi^2\} \tag{2}$$
$$K(\psi) = \mathbf{1}\{|\psi| \leq 1/2\} \tag{3}$$

*Solution*

Kernel 1: We know that the normal density integrates up to one.
Kernel 2:

$$\begin{aligned}\int_{-\infty}^{+\infty} K(\psi) &= \int_{-1}^{+1} K(\psi)\\ &= \int_{-1}^{+1} 3/4 \cdot (1-\psi^2)\,\mathrm{d}\psi\\ &= \frac{3}{4}\left[\psi - \frac{\psi^3}{3}\right]_{-1}^{1}\\ &= \frac{3}{4}\left(\frac{2}{3}+\frac{2}{3}\right)\\ &= 1\end{aligned}$$

Kernel 3:

$$\begin{aligned}\int_{-\infty}^{+\infty} K(\psi) &= \int_{-1/2}^{+1/2} K(\psi)\\ &= \int_{-1/2}^{+1/2} 1\,\mathrm{d}\psi\\ &= [\psi]_{-1/2}^{1/2}\\ &= \left(\frac{1}{2}+\frac{1}{2}\right)\\ &= 1\end{aligned}$$

*Task 2*

Show that the relation

$$\int_{-\infty}^{+\infty} \psi^k K(\psi)\, \mathrm{d}\psi = 0 \tag{4}$$

holds, for $k = 1, 3, 5, \ldots$

*Solution*

Fist we take a look at the case where $k = 1$

$$\begin{aligned}
\int_{-\infty}^{+\infty} \psi K(\psi)\, \mathrm{d}\psi &= \int_{-\infty}^{0} \psi K(\psi)\, \mathrm{d}\psi + \int_{0}^{+\infty} \psi K(\psi)\, \mathrm{d}\psi \\
&= -\int_{0}^{-\infty} \psi K(\psi)\, \mathrm{d}\psi + \int_{0}^{+\infty} \psi K(\psi)\, \mathrm{d}\psi \\
&= -\int_{0}^{-\infty} \int_{0}^{\psi} 1 \, \mathrm{d}t \, K(\psi)\, \mathrm{d}\psi + \int_{0}^{+\infty} \int_{0}^{\psi} 1 \, \mathrm{d}t \, K(\psi)\, \mathrm{d}\psi \\
&= -\int_{0}^{-\infty} \int_{t}^{-\infty} K(\psi)\, \mathrm{d}\psi \, \mathrm{d}t + \int_{0}^{+\infty} \int_{t}^{\infty} K(\psi)\, \mathrm{d}\psi \, \mathrm{d}t \\
&= -\int_{0}^{\infty} \int_{t}^{\infty} K(\psi)\, \mathrm{d}\psi \, \mathrm{d}t + \int_{0}^{+\infty} \int_{t}^{\infty} K(\psi)\, \mathrm{d}\psi \, \mathrm{d}t = 0
\end{aligned}$$

The last step is done, using the symmetry property.

*Task 3*

Proof the following Collary:

**Corallary 1** *Let assumptions DE I-IV hold. Then*

$$\mathit{plim}\, \hat{f}_x(x_0) \rightarrow f_x(x_0) \tag{5}$$

*Solution*

Proof of Collary 1:

At first: It can be show that if a random variable $x_n$ converges in mean square to a constant $c$, then it converges in probability to $c$ (See Greene 4th ed[1] p. 110, for example). This can be shown by using *Chebychev's Inequality*. Further it can be shown that if

$$\lim_{n\rightarrow\infty} \mathsf{E}\left[x_n\right] = c \text{ and } \lim_{n\rightarrow\infty} \mathsf{Var}\left[x_n\right] = 0,$$

[1] In the 5th ed p.897/898

that $x_n$ converges in mean square and hence in probability to $c$ (See also Greene p. 110/111).

So it is sufficient for showing $plim\ \hat{f}_x(x_0) \to f_x(x_0)$ that $\mathsf{E}\left[\hat{f}_x(x_0)\right] = f_x(x_0)$ and $\mathsf{Var}\left[\hat{f}_x(x_0)\right] = 0$.

$$
\begin{aligned}
\mathsf{E}\left[\hat{f}_x(x_0)\right] &= \mathsf{E}\left[\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x_i - x_0}{h}\right)\right] \\
&\overset{iid}{=} \frac{1}{h}\mathsf{E}\left[K\left(\frac{x_i - x_0}{h}\right)\right] \\
&= \frac{1}{h}\int_{-\infty}^{+\infty} K\left(\frac{\xi - x_0}{h}\right) f_x(\xi)\,\mathrm{d}\xi \\
&= \int_{-\infty}^{+\infty} K(\psi) f_x(h\psi + x_0)\,\mathrm{d}\psi \\
&= \int_{-\infty}^{+\infty} K(\psi)\left(f_x(x_0) + h\psi\partial_x f_x(x_0) + \frac{h^2\psi^2}{2}\partial_x^2 f_x(x_0) + \frac{h^3\psi^3}{6}\partial_x^3 f_x(x_0)\right)\mathrm{d}\psi \\
&= f_x(x_0) + h\partial_x^2 f_x(x_0)\int_{-\infty}^{+\infty} K(\psi)\psi^2\,\mathrm{d}\psi + \frac{h^3}{6}\int_{-\infty}^{+\infty}\partial_x^3 f_x(x_r)\psi^3 K(\psi)\,\mathrm{d}\psi \\
&= f_x(x_0) + \frac{h^2}{2}\partial_x^2 f_x(x_0)\mu_2 + \frac{h^3}{6}\int_{-\infty}^{+\infty}\partial_x^3 f_x(x_r)\psi^3 K(\psi)\,\mathrm{d}\psi \\
&= f_x(x_0) + \frac{h^2}{2}\partial_x^2 f_x(x_0)\mu_2 + o(h^2)
\end{aligned}
$$

The last step can be retraced by recognizing that:

$$
\begin{aligned}
\frac{h^3}{6}\int_{-\infty}^{+\infty}\partial_x^3 f_x(x_r)\psi^3 K(\psi)\,\mathrm{d}\psi &\le \left|\frac{h^3}{6}\int_{-\infty}^{+\infty}\partial_x^3 f_x(x_r)\psi^3 K(\psi)\,\mathrm{d}\psi\right| \\
&\le \frac{h^3}{6}\int_{-\infty}^{+\infty}\left|\partial_x^3 f_x(x_r)\psi^3 K(\psi)\right|\,\mathrm{d}\psi \\
&\le \frac{h^3}{6}\int_{-\infty}^{+\infty}|\partial_x^3 f_x(x_r)||\psi^3|K(\psi)\,\mathrm{d}\psi \\
&\le \frac{h^3}{6}b\int_{-\infty}^{+\infty}|\psi^3|K(\psi)\,\mathrm{d}\psi \\
&\le \frac{h^3}{6}bc
\end{aligned}
$$

since $|\partial_x^3 f_x(x_r)|$ is assumed to be bounded and $\int_{-\infty}^{+\infty}|\psi^3|K(\psi)\,\mathrm{d}\psi$ is also bounded.

We show now that the Variance vanishes for $n \to \infty$.

At fist the Variance can be written as

$$\begin{aligned}\mathsf{Var}\left[\hat{f}_x(x_0)\right] &= \mathsf{Var}\left[\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x_i - x_0}{h}\right)\right] \\ &= \frac{1}{(nh)^2} n \mathsf{Var}\left[K\left(\frac{x_i - x_0}{h}\right)\right] \\ &= \frac{1}{nh^2}\left(\mathsf{E}\left[K\left(\frac{x_i - x_0}{h}^2\right)\right] - \mathsf{E}\left[K\left(\frac{x_i - x_0}{h}\right)\right]^2\right).\end{aligned}$$

Therefore we have to calculate $\mathsf{E}\left[K\left(\frac{x_i - x_0}{h}^2\right)\right]$.

$$\begin{aligned}\mathsf{E}\left[K\left(\frac{x_i - x_0}{h}^2\right)\right] &= \int_{-\infty}^{\infty} K\left(\frac{\xi - x_0}{h}\right)^2 f_x(\xi)\,\mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} K(\psi)^2 f_x(\psi h + x_0) h\,\mathrm{d}\psi \\ &= h\int_{-\infty}^{\infty} K(\psi)^2 \left(f_x(x_0) + h^2 \psi \partial_x f_x(x_r)\right)\mathrm{d}\psi \\ &= h f_x(x_0) \int_{-\infty}^{\infty} K(\psi)^2\,\mathrm{d}\psi + h^2 \int_{-\infty}^{\infty} \psi K(\psi)^2 \partial_x f_x(x_r)\,\mathrm{d}\psi \\ &= h f_x(x_0) c + o(h)\end{aligned}$$

The last equality holds because of:

$$\begin{aligned}h^2 \int_{-\infty}^{\infty} \psi K(\psi)^2 \partial_x f_x(x_r)\,\mathrm{d}\psi &\leq |h^2 \int_{-\infty}^{\infty} \psi K(\psi)^2 \partial_x f_x(x_r)\,\mathrm{d}\psi| \\ &\leq h^2 \int_{-\infty}^{\infty} |\psi K(\psi)^2||\partial_x f_x(x_r)|\,\mathrm{d}\psi \\ &\leq h^2 b \int_{-\infty}^{\infty} K(\psi)^2 |\psi|\,\mathrm{d}\psi \\ &\leq h^2 b c_2 = O(h^2) = o(h)\end{aligned}$$

Now, the variance can be written as

$$\begin{aligned}\mathsf{Var}\left[\hat{f}_x(x_0)\right] &= \frac{1}{nh^2}(h f_x(x_0) c + O(h^2)) + \frac{1}{nh^2}\left(h f_x(x_0) + \frac{h^3}{2}\partial_x^2 f_x(x_0)\mu_2 + O(h^4)\right)^2 \\ &= \frac{1}{nh} f_x(x_0) c + O(n^{-1}) + \frac{f_x(x_0)}{n} + \frac{h^2}{n} f_x(x_0)\partial_x^2 f_x(x_0)\mu_2 \\ &\quad + \frac{h^4}{4n}(\partial_x^2 f_x(x_0))^2 \mu_2^2 + 2\frac{1}{nh} f_x(x_0) \cdot O(h^4) + \frac{h}{n}\partial_x^2 f_x(x_0)\mu_2 O(h^4) + O(h^6/n) \\ &= O((nh)^{-1}) + O(n^{-1}) + O(n^{-1}) + O(h^2/n) + O(h^4/n) + O(h^3/n) + O(h^5/n) + O(h^6/n) \\ &= O((nh)^{-1}) = o(1)\end{aligned}$$

*Task 4*

Show that the following theorem holds:

**Theorem 1** *Under ADE I', ADE II, ADE III' and ADE IV we obtain*

$$\mathsf{E}[\hat{f}_x(x_0)] = f_x(x_0) + \frac{h^r}{r}\mu_r \partial_x^r f_x(x_0) + o(h^r) \tag{6}$$

$$\mathsf{Var}[\hat{f}_x(x_0)] = (nh)^{-1}\kappa_0 f_x(x_0) + o((nh)^{-1}) \tag{7}$$

*Solution*

$$\begin{aligned}
\mathsf{E}\left[\hat{f}_x(x_0)\right] &= \mathsf{E}\left[\frac{1}{hn}\sum_{i=1}^{n} K\left(\frac{x_i - x_0}{h}\right)\right] \\
&= \frac{1}{hn} n \int_{-\infty}^{\infty} K\left(\frac{\xi - x_0}{h}\right) f_x(x_0)\, \mathrm{d}\xi \\
&= \frac{1}{h}\int_{-\infty}^{\infty} K(\psi)\, f_x(h\psi + x_0) h\, \mathrm{d}\psi \\
&= \int_{-\infty}^{\infty} K(\psi)\left(f_x(x_0) + \psi h \partial_x f_x(x_0) + \frac{\psi^2 h^2}{2}\partial_x^2 f_x(x_0) + \cdots\right. \\
&\quad \left. + \frac{\psi^r h^r}{r!}\partial_x^r f_x(x_0) \frac{\psi^{r+1}h^{r+1}}{(r+1)!}\partial_x^{r+1} f_x(x_{rest})\right) \mathrm{d}\psi \\
&= f_x(x_0)\int_{-\infty}^{\infty} K(\psi)\, \mathrm{d}\psi + h\partial_x f_x(x_0)\int_{-\infty}^{\infty} \psi K(\psi)\, \mathrm{d}\psi + \cdots \\
&\quad + \frac{h^r}{r!}\partial_x^r f_x(x_0)\int_{-\infty}^{\infty} K(\psi)\, \psi^r\, \mathrm{d}\psi + \frac{h^{r+1}}{(r+1)!}\int_{-\infty}^{\infty} K(\psi)\, \psi^{r+1}\partial_x^{r+1} f_x(x_{rest})\, \mathrm{d}\psi \\
&= f_x(x_0) + \frac{h^r}{r!}\partial_x^r f_x(x_0)\mu_r + \frac{h^{r+1}}{(r+1)!}\int_{-\infty}^{\infty} K(\psi)\, \psi^{r+1}\partial_x^{r+1} f_x(x_{rest})\, \mathrm{d}\psi \\
&= f_x(x_0) + \frac{h^r}{r!}\partial_x^r f_x(x_0)\mu_r + o(h^r)
\end{aligned}$$

To see why the last equation holds, we can use similar arguments like in the task before:

$$
\begin{aligned}
\frac{h^{r+1}}{(r+1)!}\int_{-\infty}^{\infty} K\left(\psi\right)\psi^{r+1}\partial_x^{r+1} f_x(x_{rest})\,\mathrm{d}\psi &\leq |\frac{h^{r+1}}{(r+1)!}\int_{-\infty}^{\infty} K\left(\psi\right)\psi^{r+1}\partial_x^{r+1} f_x(x_{rest})\,\mathrm{d}\psi| \\
&\leq \frac{h^{r+1}}{(r+1)!}\int_{-\infty}^{\infty} |K\left(\psi\right)\psi^{r+1}\partial_x^{r+1} f_x(x_{rest})|\,\mathrm{d}\psi \\
&\leq \frac{h^{r+1}}{(r+1)!}\int_{-\infty}^{\infty} K\left(\psi\right)|\psi^{r+1}||\partial_x^{r+1} f_x(x_{rest})|\,\mathrm{d}\psi \\
&\leq \frac{h^{r+1}}{(r+1)!} b\int_{-\infty}^{\infty} K\left(\psi\right)|\psi^{r+1}|\,\mathrm{d}\psi \\
&\leq \frac{h^{r+1}}{(r+1)!} bc = O(h^{r+1}) = o(h^r)
\end{aligned}
$$

Since the $r+1^{th}$ derivative of $f_x(x)$ is assumed to be bounded and the same holds for the reminding integral term.

Variance is the same as in the case of the usual kernels.