Advanced Mathematical Methods

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1 Linear Algebra

Dr. Julie Schnaitmann

Department of Statistics, Econometrics and Empirical Economics

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Wirtschafts- und Sozialwissenschaftliche Fakultät

Outline: Linear Algebra

Readings

- Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne Strøm. *Further Mathematics for Economic Analysis*. Prentice Hall, 2008 Chapter 1

Online Resources

MIT course on Linear Algebra (by Gilbert Strang)

- Lecture 21: Eigenvalues and Eigenvectors
  https://www.youtube.com/watch?v=lXNXrLcoerU
- Lecture 22: Powers of a square matrix and Diagonalization
  https://www.youtube.com/watch?v=13r9QY6cmjc
- Lecture 26: Symmetric matrices and positive definiteness
  https://www.youtube.com/watch?v=umt6BB1nJ4w
- Lecture 27: Positive definite matrices and minima – Quadratic forms
  https://www.youtube.com/watch?v=vF7eyJ2g3kU

1.8 Eigenvalues and eigenvectors

Assume a scalar λ exists such that

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

λ: eigenvalue
$\boldsymbol{x}$: eigenvector

Find λ via the homogenous linear equation system

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = 0$$

1.8 Eigenvalues and eigenvectors

The properties of a quadratic homogenous linear equation system imply that:

- in any case a solution does exist;
- if $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) \neq 0$, then $\bar{\boldsymbol{x}} = 0$ is the trivial solution;
- only if $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$ there is a non-trivial solution.

1.8 Eigenvalues and eigenvectors

Determination of the eigenvalues via the *characteristic equation*:

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| \;=\; 0 \quad\Longleftrightarrow\quad (-1)^n\lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0 \;=\; 0$$

for every (real or complex) eigenvalue λ_i of the $(n \times n)$-Matrix $\boldsymbol{A}$ we can calculate the respective eigenvector $\boldsymbol{x}_i \neq 0$ solving the homogenous linear equation system

$$(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{x}_i = 0\,. \tag{1}$$

The properties of homogenous linear equation systems imply that the solution of eq. (1) is not unambiguous, i.e. for the eigenvalue λ_i we can find infinitely many eigenvectors $\boldsymbol{x}_i$.

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1.8 Eigenvalues and eigenvectors

$\boldsymbol{A}$ und $\boldsymbol{B}$ (quadratic matrices of order n) are similar if a regular $(n \times n)$ - matrix $\boldsymbol{C}$ exists, such that

$$\boldsymbol{B} = \boldsymbol{C}^{-1} \boldsymbol{A} \, \boldsymbol{C} \; .$$

Special case: **symmetric matrices**

For a symmetric $(n \times n)$-matrix $\boldsymbol{A}$ it holds that the normalized eigenvectors $\tilde{\boldsymbol{x}}_j$ with $j = 1, \ldots, n$ have the property

1. $\tilde{\boldsymbol{x}}_j' \tilde{\boldsymbol{x}}_j = 1$ for all j and
2. $\tilde{\boldsymbol{x}}_i' \tilde{\boldsymbol{x}}_j = 0$ for all $i \neq j$.

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Principle axis theorem

collecting the normalized eigenvectors $\tilde{\boldsymbol{x}}_j$ $(j = 1, \ldots, n)$ in a new matrix $\boldsymbol{T} = [\tilde{\boldsymbol{x}}_1 \cdots \tilde{\boldsymbol{x}}_n]$ with the property $\boldsymbol{T}^{-1} = \boldsymbol{T}'$ yields the diagonalization of $\boldsymbol{A}$ as follows:

$$\boldsymbol{D} = \boldsymbol{T}'\boldsymbol{A}\boldsymbol{T} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix}.$$

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

1) The product of the eigenvalues of a $n \times n$ matrix yields its determinant: $|\boldsymbol{A}| = \prod_{i=1}^{n} \lambda_i$.
2) From 1.) it follows that a singular matrix must have at least one eigenvalue $\lambda_i = 0$.
3) The matrices $\boldsymbol{A}$ and $\boldsymbol{A}'$ have the same eigenvalues.
4) For a non-singular matrix $\boldsymbol{A}$ with eigenvalues λ we have: $|\boldsymbol{A}^{-1} - \frac{1}{\lambda}\boldsymbol{I}| = 0$.
5) Symmetric matrices have only real eigenvalues and orthogonal eigenvectors.

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1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

6) The rank of a symmetric matrix $\boldsymbol{A}$ is equal to the number of eigenvalues different from zero.
7) The sum of the eigenvalues is equal to the trace: $\text{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} \lambda_i$.
8) It holds that the eigenvalues of $\boldsymbol{A}^k$ are λ_i^k for all $i = 1, \dots, n$ as $\boldsymbol{A}^k = \boldsymbol{T}\boldsymbol{\Lambda}^k\boldsymbol{T}^{-1}$.
9) $\boldsymbol{A}$ has n independent eigenvectors and is diagonalizable if all eigenvalues λ_i are distinct.

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1.9 Quadratic forms and sign definitness

Definitions

- Degree of a polynomial
- Form of nth degree
- special case: quadratic form

$$Q(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

1.9 Quadratic forms and sign definitness

A quadratic form $Q(x_1, x_2)$ for two variables x_1 and x_2 is defined as

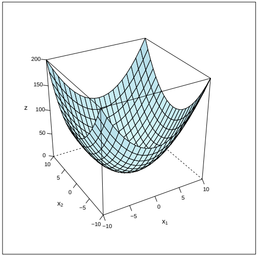
$$Q(x_1, x_2) = \underset{(1\times2)(2\times2)(2\times1)}{\mathrm{x}' \; \mathrm{A} \; \mathrm{x}} = \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}\, x_i\, x_j$$

where $a_{ij} = a_{ji}$ and, thus,

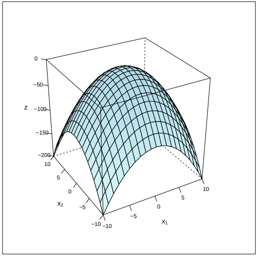
with the symmetric coefficient matrix $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$.

1.9 Quadratic forms and sign definitness

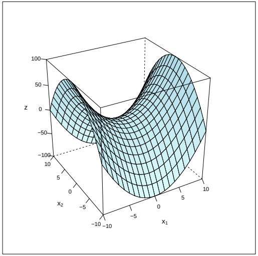
Graph of the positive definite form $Q(x_1, x_2) = x_1^2 + x_2^2$



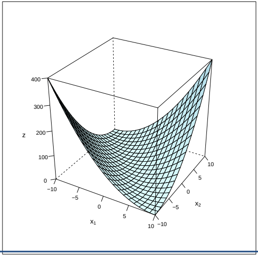
Graph of the negative definite form $Q(x_1, x_2) = -x_1^2 - x_2^2$



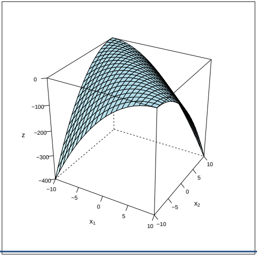
Graph of the indefinite form $Q(x_1, x_2) = x_1^2 - x_2^2$



Graph of the positive semidefinite form $Q(x_1, x_2) = (x_1 + x_2)^2$



Graph of the negative semidefinite form $Q(x_1, x_2) = -(x_1 + x_2)^2$



1.9 Quadratic forms and sign definitness

The quadratic form associated with the matrix A (and thus the matrix A itself) is said to be

positive definite,	if $Q = x'Ax > 0$	for all $x \neq 0$
positive semi-definite,	if $Q = x'Ax \geq 0$	for all x
negative definite,	if $Q = x'Ax < 0$	for all $x \neq 0$
negative semi-definite,	if $Q = x'Ax \leq 0$	for all x

Otherwise the quadratic form is **indefinite**.

Note: For any quadratic matrix A it holds that $x'Ax = x'Bx$ with $B = 0,5 \cdot (A + A')$, a symmetric matrix.

1.9 Quadratic forms and sign definitness

The quadratic form associated with the matrix A (and thus the matrix A itself) is said to be

positive definite,	if $Q = \mathrm{x}'\mathrm{A}\mathrm{x} > 0$	for all $\mathrm{x} \neq 0$
positive semi-definite,	if $Q = \mathrm{x}'\mathrm{A}\mathrm{x} \geq 0$	for all x
negative definite,	if $Q = \mathrm{x}'\mathrm{A}\mathrm{x} < 0$	for all $\mathrm{x} \neq 0$
negative semi-definite,	if $Q = \mathrm{x}'\mathrm{A}\mathrm{x} \leq 0$	for all x

Otherwise the quadratic form is **indefinite**.

Note: For any quadratic matrix A it holds that $\mathrm{x}'\mathrm{A}\mathrm{x} = \mathrm{x}'\mathrm{B}\mathrm{x}$ with $\mathrm{B} = 0,5 \cdot (\mathrm{A} + \mathrm{A}')$, a symmetric matrix.

1.9 Quadratic forms and sign definitness

The quadratic form $Q(\boldsymbol{x})$ is

- positive (negative) definite, if **all** eigenvalues of the matrix A are positive (negative): $\lambda_j > 0$ ($\lambda_j < 0$) $\forall j = 1, 2, \ldots, n$;
- positive (negative) semi-definite, if **all** eigenvalues of the matrix A are non-negative (non-positive): $\lambda_j \geq 0$ ($\lambda_j \leq 0$) $\forall j = 1, 2, \ldots, n$ and **at least one** eigenvalue is equal to zero;
- indefinite, if two eigenvalues have different signs.

1.9 Quadratic forms and sign definitness

Properties of positive definite and positive semi-definite matrices

1) Diagonal elements of a positive definite matrix are strictly positive. Diagonal elements of a positive semi-definite matrix are nonnegative.
2) If A is positive definite, then A^{-1} exists and is positive definite.
3) If X is $n \times k$, then $X'X$ and XX' are positive semi-definite.
4) If X is $n \times k$ and $rk(X) = k$, then $X'X$ is positive definite (and therefore non-singular).

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