Advanced Time Series

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Overview of today's exercise

- Programming rules in GAUSS
- Drawing random variables
- Simulation of an AR(1) process

Programming rules in GAUSS

- ALWAYS start simple!!! Start with the simple calculations! Next enrich your program step by step!
- Start in a procedure ALWAYS from the inside and then go outside!
- CHECK frequently the results of your programming! (check them in the output window.)
- ALWAYS start with small n!

Programming rules in GAUSS 2

- Comment your program!!!
- Use useful and sensible names for your variables and programs!
- Create your own program collection!

Overview of today's exercise

- Programming rules in GAUSS
- Writing the log-likelihood function AR(1) MA(1) - Theory
- Using CML in GAUSS
- Estimating the parameters of a MA(1) process
- Estimating the parameters of a AR(1) process

Likelihood function AR(1) for 1st and 2nd observation

$$\begin{aligned} Y_t &= c + \phi Y_{t-1} + \varepsilon_t \\ \varepsilon_t &= iidN(0, \sigma^2) \end{aligned}$$

density of first observation

$$\begin{aligned} Y_1 &\sim N(c/(1-\phi), \sigma^2/(1-\phi^2)) \\ f_{Y_1}(y_1; c, \phi, \sigma^2) &= \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2/(1-\phi^2)}} \exp\left[\frac{-\{y_1 - [c/(1-\phi)]\}^2}{2\sigma^2/(1-\phi^2)}\right] \end{aligned}$$

density of second observation

$$\begin{aligned}
Y_2 &= c + \phi Y_1 + \varepsilon_2 \\
(Y_2 | Y_1 = y_1) &\sim N((c + \phi y_1), \sigma^2) \\
f_{Y_2|Y_1}(y_2|y_1;\theta) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(y_2 - c - \phi y_1)^2}{2\sigma^2}\right] \\
\varepsilon_2 &= y_2 - c - \phi y_1 \\
f_{Y_2|Y_1}(y_2|y_1;\theta) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(\varepsilon_2)^2}{2\sigma^2}\right]
\end{aligned}$$

joint density of first and second observation

$$f_{Y_2,Y_1}(y_2, y_1;\theta) = f_{Y_1}(y_1;\theta) \cdot f_{Y_2|Y_1}(y_2|y_1;\theta)$$

Writing the joint likelihood function AR(1)

$$f_{Y_T,Y_{T-1},\dots,Y_1}(y_T, y_{T-1}, \dots, y_1; \theta) = f_{Y_1}(y_1; \theta) \cdot \prod_{t=2}^{T} f_{Y_t|Y_{t-1}}(y_t|y_{t-1}; \theta)$$

$$f_{Y_t|Y_{t-1}}(y_t|y_{t-1}; \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(y_t - c - \phi y_{t-1})^2}{2\sigma^2}\right]$$

$$f_{Y_t|Y_{t-1}}(y_t|y_{t-1}; \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(\varepsilon_t)^2}{2\sigma^2}\right]$$

Taking logs yields

$$\mathscr{L}(\theta) = \log f_{Y_1}(y_1; \theta) + \sum_{t=2}^{T} \log f_{Y_t|Y_{t-1}}(y_t|y_{t-1}; \theta)$$

Writing the log-likelihood function AR(1)

$$\begin{aligned}
\mathscr{L}(\theta) &= \log f_{Y_1}(y_1;\theta) \rightarrow \text{deterministic} \\
&+ \log f_{Y_2|Y_1}(y_2|y_1;\theta) \rightarrow \log\left((\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(\varepsilon_2)^2}{2\sigma^2}\right]\right) \\
&\ldots \\
&+ \log f_{Y_T|Y_{T-1}}(y_T|y_{T-1};\theta) \rightarrow \log\left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(\varepsilon_T)^2}{2\sigma^2}\right]\right)
\end{aligned}$$

MA(1) Process

$$
\begin{aligned}
Y_t &= \mu + \theta\varepsilon_{t-1} + \varepsilon_t \\
\varepsilon_t &= iidN(0, \sigma^2)
\end{aligned}
$$

Writing the likelihood function MA(1)

$$
\begin{aligned}
Y_t|_{\varepsilon_{t-1}} &\sim N((\mu + \theta\varepsilon_{t-1}), \sigma^2) \\
f_{Y_t|\varepsilon_{t-1}}(y_t|\varepsilon_{t-1};\theta) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(y_t - \mu - \theta\varepsilon_{t-1})^2}{2\sigma^2}\right] \\
\varepsilon_t &= y_t - \mu - \theta\varepsilon_{t-1} \\
f_{Y_t|\varepsilon_{t-1}}(y_t|\varepsilon_{t-1};\theta) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(\varepsilon_t)^2}{2\sigma^2}\right]
\end{aligned}
$$

Log likelihood function MA(1)

$$\mathscr{L}(\theta) =$$

$$\log f_{Y_T, Y_{T-1}, \ldots . Y_1 | \varepsilon_0 = 0}(y_T, y_{T-1}, \ldots, y_1 | \varepsilon_0 = 0; \theta) =$$

$$-T \log(\sqrt{2\pi\sigma^2}) - \sum_{t=1}^{T} \frac{\varepsilon_t^2}{2\sigma^2}$$

How to get ε_t? - Recursion

$$\begin{aligned}
\varepsilon_1 &= y_1 - \mu \quad \text{with} \quad \varepsilon_0 = 0 \\
\varepsilon_2 &= y_2 - \mu - \theta\varepsilon_1 \\
&\dots \\
\varepsilon_T &= y_T - \mu - \theta\varepsilon_{T-1}
\end{aligned}$$

Writing the conditional log-likelihood function MA(1)

$$
\begin{aligned}
\mathscr{L}(\theta) &= f_{Y_1|\varepsilon_0=0}(\cdot;\theta) \to \log\left((\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(\varepsilon_1)^2}{2\sigma^2}\right]\right) \\
&+ \log f_{Y_2|Y_1,\varepsilon_0=0}(\cdot;\theta) \to \log\left((\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(\varepsilon_2)^2}{2\sigma^2}\right]\right) \\
&\quad ... \\
&+ \log f_{Y_T|Y_{T-1},\dots,Y_1,\varepsilon_0=0}(\cdot;\theta) \to \log\left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(\varepsilon_T)^2}{2\sigma^2}\right]\right)
\end{aligned}
$$

CML procedure

- Numerical optimization of a function using an algorithm
- Input: function to be minimized and starting values for parameters, and data
- Output: vector of parameters and function value at minimum

CML procedure-CALL

{ x,f,g,cov,retcode } = CML(dataset,vars,&fct,start)

INPUT

dataset - name of data matrix stored in memory

vars - character vector of labels selected for analysis

take vars = 0;

fct - the name of a procedure that returns the log-likelihood,

e.g. &malikeliproc

start - a Kx1 vector of start values

CML procedure-CALL

{ x,f,g,cov,retcode } = CML(dataset,vars,&fct,start)

OUTPUT

x - Kx1 vector, estimated parameters

f - scalar, function at minimum (mean log-likelihood)

g - Kx1 vector, gradient evaluated at x

cov - KxK matrix, covariance matrix of the parameters

retcode - scalar, return code

CML procedure-GLOBALS

Example:

```
_cml_Algorithm=1;
_cml_LineSearch=1;
_cml_DirTol = 1e-5;
_cml_CovPar_=1;
```

CML Global variables I

CML global: `_cml_DirTol`=0.000000001;

`_cml_DirTol` = scalar is a convergence tolerance for gradient of estimated coefficients.

Default = 1e-5.

When this criterion has been satisfied CML will exit the iterations.

Important!!

Some applications demand a small value in order to prevent convergence on a local minimum!!!!! (local vs. global optima)

CML Global variables II

CML global:`_cml_Algorithm`= scalar indicator for optimization method

`_cml_Algorithm`

= 1, BFGS (Broyden, Fletcher, Goldfarb, Shanno)

= 2, DFP (Davidon, Fletcher, Powell)

= 3, NEWTON (Newton-Raphson)

= 4, BHHH

CML Global variables III

`_cml_LineSearch`;

= 1 One

= 2, STEPBT (default)

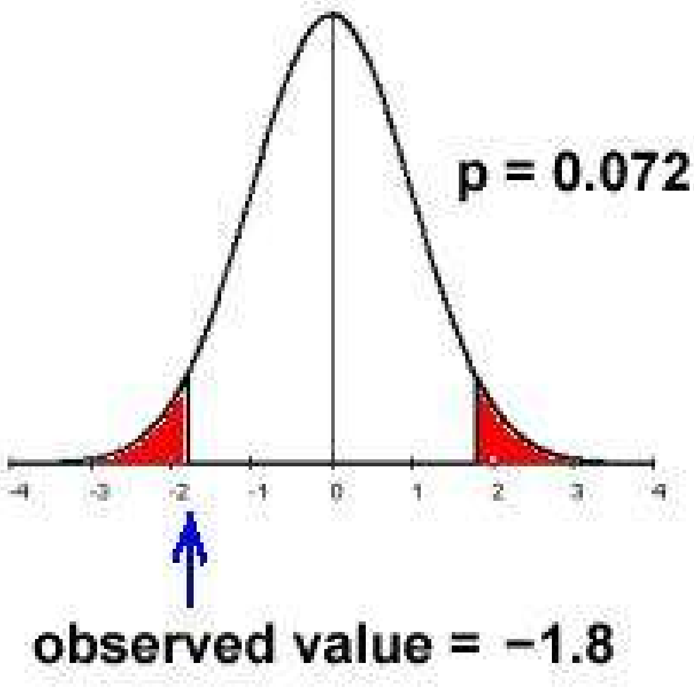
= 3, HALF (step-halving)

= 4, BRENT

= 5, BHHHSTEP

Reminder: P-value

Two Sided Test: Example



Reminder: P-value

```
se = sqrt(diag(cov));

t_test = thx./se;

p_val = 2*cdftc(t_test,rows(y)-rows(thx));
```

General:

One-sided `p_val = cdftc(teststat,degrees of freedom)`

Two-sided `p_val = 2*cdftc (teststat, degrees of freedom)`

Reminder: Including Source Files

To render your programs less confusing procedures can be written into source files.

The source file is then included into the program using:

#include mysourcefile.src;

A source file consists only of procedure code, no *hard* code should be written into a source file;

Testing for Stationarity

Dickey-Fuller Unit Root Test

$$y_t = \rho y_{t-1} + \varepsilon_t$$

- Stationarity Test
- Null hypothesis: unit root (non stationarity), i.e. that $\rho = 1$

Simulation of Dickey-Fuller Test Statistic

- Non-standard asymptotic distribution of Unit Root processes
- Inference requires simulation of asymptotic distribution
- Asymptotic distribution depends on specification of the true process (constant, time trend)

Case 1

1. Simulate a Random Walk

$$\text{True Process: } y_t = \phi y_{t-1} + \varepsilon_t$$

2. Conduct an OLS regression

$$\text{Estimated Process: } y_t = \rho y_{t-1} + \varepsilon_t,$$

calculate the t-statistic for the null hypothesis that the true value of ρ equals 1

3. Simulate the test statistic

Run Step 2 n=10000 times and sort the t-values into quantiles

Estimation of a GARCH(1,1)

Stylized facts of financial return data

A simple model to account for these stylized facts

mean equation:(y_t is a log return time series)

$$y_t = c + \varepsilon_t$$

$$\varepsilon_t = y_t - c$$

variance equation

$$h_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}$$

Writing the conditional likelihood function GARCH(1,1)

$$f(y_t|y_{t-1},....y_0;\theta) = \frac{1}{\sqrt{2\pi h_t}}\exp\left[\frac{-(y_t-c)^2}{2h_t}\right]$$

Conditional log likelihood function:

$$\begin{aligned}\mathscr{L}(\theta) &= \sum_{t=1}^{T}\log f(y_t|y_{t-1},....y_0;\theta)\\ &= -\frac{1}{2}\log(2\pi)-\frac{1}{2}\sum_{t=1}^{T}\log(h_t)-\frac{1}{2}\sum_{t=1}^{T}\frac{(y_t-c)^2}{h_t}\end{aligned}$$