Comments on the Contributions

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Abstract The contributions to this volume represent a broad range of aspects of proof-theoretic semantics. Some do so in the narrower, and some in the wider sense of the term. Some deal with issues I have been concerned with directly, and some tackle further problems. All of them open interesting new perspectives and develop the field in different directions. I will briefly comment on the significance of each contribution here.

Sundholm on Frege's anticipation of the deduction theorem. Two papers are historically oriented: Göran Sundholm's and to some extent that by Neil Tennant (which will be discussed later). Both deal with Frege. I welcome this very much, not only as Frege has been a main topic in my own historical studies, but also because Frege's work is of high systematic relevance to proof-theoretic semantics. He was the first to develop a precise notion of logical deduction, based on the idea that we proceed from judgements already seen as true to new true judgement in a gap-free way. Even the sentences of his concept script (*Begriffsschrift*) can be read as a two-dimensional notation of sequents, so that his proofs proceed in a sort of sequent calculus (Schroeder-Heister 1997; 2014). Sundholm draws attention to the much neglected logical content of Section 17 of Frege's *Foundations of Arithmetic* (1884). He shows that Frege in a sense anticipates the deduction theorem as well as its proof in Hilbert-Bernays (1934). In pointing this out, he discusses Frege's notion of analyticity in detail, as the conditional statements generated by the deduction theorem represent analytical judgements. What I find particularly interesting, besides the wealth of discussion of related concepts such as axiom, tautology, self-evidence, topic-neutrality and the like, is the fact that Sundholm, at least implicitly, alludes to the fact that in Frege there is a notion of proof from assumptions, in addition to proofs of hypothetical judgements, because otherwise talking of the deduction

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theorem does not make much sense.[1] Though there are several places, where Frege speaks against assumptions as a special mode of judgement (e.g., Frege 1906; 1923), in the passage discussed by Sundholm he speaks of a deduction which starts from a *fact* ("Tatsache"), that is, something that comes from outside, and argues that the content of this deduction can be expressed by an (analytical) hypothetical judgement. The judgement we would call an assumption (as the starting point of the deduction), figures in Frege as the statement of an external fact. Formally, for the validity of the subsequent chain of deduction, it does not make any difference whether we start from an external fact or from an internally specified assumption: we have the idea of something from which a logical deduction starts, and which can then become ingredient of the deduction theorem.[2]

Tennant on Frege's class theory and the logic of sets. Tennant argues that Frege's fundamental assumptions would have led to a system corresponding to Core Logic, the foundational system for logic and mathematics Tennant has advocated and developed for several decades. Core Logic is itself very interesting from the standpoint of proof-theoretic semantics, as it relies on specific inferentialist assumptions concerning harmony, normal form etc. What makes it particularly significant is that is includes and accounts for crucial issues beyond standard proof-theoretic semantics, such as the notion of relevance as well as the non-insistance on transitivity (or cut in the sequent calculus). The latter is a point I have also put forward in several contexts, in particular in the development of definitional reflection and its application to paradoxes, a field where Tennant and myself have similar intuitions (Tennant, 1982, Schroeder-Heister, 2012b)[3]. What is also important about Core Logic is that it is a free logic, a topic underdeveloped in standard proof-theoretic semantics (or only discussed in connection with the denotation of proofs, but not in connection with singular terms), as the whole area of denotation is not given the attention it deserves. If we base our theorizing not on denotation and truth but on inference and proof as basic concepts, we must be able to say how we can incorporate the denotational concepts. Core Logic is perhaps the best worked-out system in this direction, apart from constructive type theories in the Martin-Löf tradition. Though itself set-theoretic in spirit, with the latter it has in common that it introduces the terms and predicates needed through particular rules. In standard approaches to set theory, the existence of certain sets is postulated axiomatically, but the terms denoting these sets are not primitive symbols of the language itself but have a status similar to eliminable definite descriptions. Tennant calls them "pasigraphs" as they are used throughout, even in

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[1] When reading Frege-implications as sequents, an iterated implication with the antecedents ("Unterglieder") A and B and the succedent C ("Oberglied") cannot be distinguished from one with the antecedent A and succedent B-*implies*-C — it is just a different way of looking at the sequent, which means that a deduction theorem in this sense is built into Frege's notation (see Schroeder-Heister 1999; 2014).

[2] Frege obviously means *empirical facts* as he is making his remarks in the context of the discussion of induction, though his logical claim is completely independent of this context. His deduction theorem also holds when a deduction starts, for example, from a tautology or a contradiction.

[3] Even though concerning Ekman's paradox, our opinions diverge; see Schroeder-Heister and Tranchini (2017; 2021) and Tennant (2021).

the standard expositions, and insists that they be inferentially defined by introduction and elimination inferences.

Prawitz on validity of inference and argument. Proof-theoretic validity lies at the core of proof-theoretic semantics. Whether this programme will be successful depends in many respects on whether it can provide notions of validity that can compete with what is available in model-theoretic semantics. Defining a promising concept has been one of the central philosophical occupations of Prawitz's research after his groundbreaking work on natural deduction (Prawitz, 1965). The programme of defining validity has turned out more difficult than perhaps envisaged at the beginning of the 1970s. For a long time Prawitz has advocated a notion of validity of arguments or proofs, on which, as a secondary notion, a notion of validity of inferences could build (Prawitz 1971; 1973; 2006). I have myself tried to formally explicate this programme (Schroeder-Heister, 2006), though I have always tended to give the validity of inferences a conceptually primary status. For somewhat different reasons Prawitz now also favours such an approach. The big problem, however, is that the notions of validity of inference and validity of proof interact. If one wants to put validity of inferences first and that of proofs second — the validity of proofs as depending on that of the inferences involved in the proofs — there is the problem that there needs to be some property whose transmission from premisses to conclusion of an inference establishes its validity. This property would be some sort of validity of proofs (in the model-theoretic case it is the notion of truth in a model). Prawitz and myself are both working on hopefully resolving this prima facie circularity. His contribution to this volume can be seen as an intermediary step towards a solution. It characterizes the notions of inference, validity, argument, canonicality etc. together with their interdependencies, without yet trying to bring these concepts into a well-founded order. This is a step forward, as it sets up certain adequacy conditions these concepts have to obey. Currently it is not clear that in the end a well-founded order of concepts can be established. Perhaps the structural characterization of concepts and their interrelations is the best we can achieve: Proof-theoretic validity of inferences and arguments would then constitute a structure that can materialize in different ways. As this is work still under way, I leave the discussion here at this general level, and just stress its fundamental character.

De Campos Sanz on Kolmogorov and the theory of problems. Starting from Kolmogorov's interpretation of logical propositions as problems for which a solution is sought, de Campos Sanz arrives at a couple of distinctions that are directly relevant to proof-theoretic semantics. He presents a "reduction semantics", according to which the consequence from A to B is interpreted as the reduction of the problem B to the problem A, and proves its adequacy with respect to intuitionistic logic. Applying this apparatus to problem solving through constructions in elementary geometry leads him to consider additional logical constants, an example of which is what he calls "before-after conjunction", which is intended to capture the order in which constructions are carried out. According to de Campos Sanz it cannot be defined in terms of standard logical constants but should nevertheless be conceived as logical. He is even led to consider, besides the notion of *assumption* in the sense that a solution to a problem is supposed to be available, a more general notion of *hypothesis* in *reductio proofs* (which are abundant in geometry). From my point of view, Kolmogorov's *problem interpretation* of logic should be considered in connection with reductive approaches such as tableau systems and dialogical interpretations — in this sense de Campos Sanz's usage of the term "reduction semantics" is highly accurate. At a more general level, investigations such as de Campos Sanz's show us that other forms of argumentation together with other sorts of concepts come into play when we leave the narrow path of standard logical systems. This opens perspectives which are still way beyond the grasp of proof-theoretic semantics as it stands now.

Pereira, Haeusler and Nascimento on disjunctive syllogism and *ex falso*. The paper by Pereira, Haeusler and Nascimento presents two systems between minimal and intuitionistic logic in which *ex falso quodlibet* is not a consequence of disjunctive syllogism. This is a significant contribution to a discussion in logic since antiquity, as disjunctive syllogism is considered a *prima facie* plausible rule, whereas the *ex falso* principle does not have this degree of plausibility. The first system is a variant of Tennant's intuitionistic relevance logic, but without the relevance restriction concerning assumptions, the second is an adaptation of an intuitionistic multiple-conclusion system (developed by the authors in earlier work) to a single conclusion system, where the visibility of assumptions is restricted in a certain way. This is immediately relevant to proof-theoretic semantics as it gives us novel options of framing deductive reasoning which go beyond standard natural deduction (as the formal paradigm of proof-theoretic semantics). The first option, which the paper shares with Tennant, gives the absurdity constant a kind of structural role, when it occurs in generalized elimination inferences. The second option and its associated notion of "visibility" gives an intuitively more perspicuous rendering of what otherwise would be achieved by a not-so-perspicuous multiple conclusion system. *Visibility* might perhaps be considered to be an additional basic concept in the definition of a proof structure, which certainly can be extended to the general case of arbitrary n-ary connectives and even to general reasoning with inductively defined objects. The paper also shows that intuitionistic and classical logic are not the only logical systems to be discussed, and that an intermediate system may be more faithful to our semantical intuitions. This corresponds to observations also made in Piecha and Schroeder-Heister (2019) that intuitionistic logic is not the logic of standard proof-theoretic validity, and that a logic weaker than classical but stronger than intuitionistic logic might be taken into consideration.

Indrzejczak on the logicality of equality. In addition to being an encompassing overview of many treatments of equality in natural deduction and in the sequent calculus, this paper shows that equality can indeed be considered a logical constant in the sense of Došen's (1989) proposal using double line rules. However, Indrzejczak does not take over Došen's own proposal concerning equality rules, which for Indrzejczak is of a 'global' nature, since rephrasing it's rules in natural deduction would need a rewriting of a whole proof. He prefers a 'local' variant with 'free' [my terminology] predicate constants obeying the side condition that they do not occur otherwise in the sequent. I would consider this a hidden second-order treatment of equality which effectively expresses Leibniz equality. This shows the strength of Leibniz's concept if one wants to obtain a 'local' notion. It also points to the fact that the relationship between first- and second-order concepts is a critical topic of proof-theoretic semantics[4], which is far from being settled and which requires further research. At the same time Indrzejczak points to the unclear relationship between logicality and semantics. Even though many (such as Došen and also myself in several publications, e.g., Schroeder-Heister, 1984b) have tried to keep these two issues apart, the criteria for logicality, when formulated in terms of inference rules, are often of a kind that they can be read as meaning conferring and thus as semantical conditions.

Arndt on rules for implication elimination. Arndt discusses different ways of formulating the implication elimination rule in natural deduction. In an earlier publication (Arndt, 2019), he distinguished eight possible variants of implication rules in the sequent calculus. They differed in the way they could be derived, by means of the cut rule, from an axiomatic sequent expressing *modus ponens*. These variants included the standard implication-left rule as well as my proposal for a revised implication-left rule (Schroeder-Heister, 2011). Here this is carried over to natural deduction, where the role of cut is now taken by a rule of explicit composition (which is related to explicit substitution, see Abadi, Cardelli, Curien, and Lévy, 1991; Arndt and Tesconi, 2014). Furthermore, a notational device is added that forces certain premisses of rules to be assumptions (to "stand proud" in Tennant's (1992; 2002) terminology). The result is a congruence between natural deduction and the sequent calculus which puts them in much closer parallel than the usual translations between these types of systems. At the same time it gives a proper understanding of bidirectionality in natural deduction, which I had proposed (at least programmatically) in Schroeder-Heister (2009). Although these are observations at the syntactical level of formal systems, they are highly relevant to proof-theoretic semantics, as the way reasoning is framed depends on which options we have in unterstanding rules, assumptions, proof composition and the like. This is the first investigation I am aware of, which takes subtle differences in the formulations of inference rules into account by making explicit the possible differences in the status of their premisses.

Liang and Miller on focusing Gentzen's LK. The contribution of Liang and Miller gives, from the perspective of proof search, a presentation of the authors' "focused" sequent calculus which overcomes weaknesses of Gentzen's classical sequent calculus LK. It is well-known that the reduction procedures for LK, as applied, for example, in cut elimination proofs, are not very deterministic due to the fact that a great number of permutations between inferences are possible. This makes the sequent calculus differ from the calculus of natural deduction, which, however, has other deficiencies when it comes to proof search. In order to keep the advantages of the sequent calculus for computational purposes, it is here expanded to a system which is syntactically more involved by considering atoms and connectives of different polarities and, correspondingly, two different sequent arrows. One is compensated for this by a more streamlined and more deterministic structure of proofs which is not

[4] See also, in another context, the contribution by Pistone and Tranchini (2023, this volume).

only useful for computational purposes but also for demonstrations of metalogical results such as Herbrand's theorem. It generalizes systems of a related kind developed by Girard, Andreoli and others. This paper shows that the area of proof search, which at the time of the origin of proof-theoretic semantics was much neglected, is becoming an integral part of it in the sense that proof-search aspects are built into the semantically-understood inferences.

Pistone and Tranchini on intensional harmony as isomorphism. The concept of harmony is central in the proof-theoretic semantics of natural deduction. That the consequences of elimination rules of a logical sign have the same deductive power as the conditions of its introduction rules is often seen as a justification of these rules. Harmony also gives rise to identities between proofs in the sense that certain successions of introductions and eliminations can be "reduced" or "contracted", yielding a proof which is still considered the same as the original one. In this sense, harmony is the basis of an intensional proof-theoretic semantics based on the notion of identity of proofs. Because a general definition of harmony was a desideratum, in Schroeder-Heister (2015) I developed an approach which translated the conditions of introduction rules as well as consequences of eliminations rules into formulas of second-order logic with propositional quantification and defined harmony as logical equivalence of these translations. Pistone and Tranchini point out that equivalence is too weak a notion for an appropriate intensional notion of harmony and that some sort of isomorphism of these translations is needed. This is not available in the metatheory of second-order logic based on beta and eta reduction. As a solution, the authors propose an additional so-called "epsilon reduction" which is based on the idea that there is exactly one proof of polymorphic identity in second-order logic. This is a major step forward beyond what I had proposed, as for many cases it leads to a plausible notion of harmony. It also demonstrates the significance of second-order propositional logic for the proof-theoretic semantics of elementary propositional logic.

Wansing on synonymy. The intensional notion of proof identity induces a notion of isomorphism between propositions. One would consider A and B isomorphic to one another if there are proofs of B from A and of A from B, such that the composition of these proofs yields the identity proof of A from A and of B from B. "Yields" here means that the given composition of proofs is identical (in the sense of proof identity) to the identity proof. Wansing, who is working in a bilateral framework of proofs and disproofs, gives a different definition of synonymy (his term for isomorphism). For him A and B are synonymous, if there are identical proofs from A to B and from B to A, as well as identical disproofs between these propositions. His notion of proof identity does not require that the propositions proved (and assumed) are the same between identical proofs, which is against a principal *tenet* in standard intensional proof-theoretic semantics. Identity in Wansing's sense is defined by a structural correspondence of sequent-style proofs. This definitely gives novel incentives to the discussion on the identity of proofs, both from the structural point of view (identical proofs for different propositions), but also from the consideration of the sequent calculus and the appeal to bilateralism, which are very much neglected in current discussions of proof identity. So far, only the discussion of the principal type of a lambda term as representing the structure of a proof comes close to the idea that there can be identical proofs (proofs of identical structure) of different theorems (see Hindley, 1997; Rezende de Castro Alves, 2019). In the discussion of identity of proofs and thus in intensional proof-theoretic semantics (see Tranchini, 2023), there are many conceptual aspects still open, and Wansing is providing a fresh look at some of these.

Kahle and Santos on paradoxes. Kahle and Santos discuss the relationship between the conceptual constructions of logical, semantical and set-theoretic paradoxes and the logic used to derive a contradiction. It is the logic which renders these constructions paradoxical. On the other hand, it is extremely difficult to make specific logical features responsible for the paradoxical outcome — just passing to another logical system, for example from classical logic to a logic without excluded middle such as intuitionistic logic does not change the situation. What changes it are global considerations such as normalization requirements in the sense of Prawitz (1965, Appendix B) and Tennant (1982). Therefore Kahle and Santos plead to further scrutinize the conceptual constructions of the paradoxes, but from a consequentialist point of view, for which I have argued myself (based on ideas of Lars Hallnäs): not restricting conceptual definitions themselves, but classifying definitions according to possible consequences including the non-eliminability of cut etc. (Schroeder-Heister, 2012b). However, while Kahle and Santos discard the reference to substructural logics to avoid the paradoxes and criticize some attempts I made in this direction (Schroeder-Heister 2012a; 2016), I see an option that limits the rule of contraction. This is a limitation not in the global substructural, but in a local intensional sense (see Schroeder-Heister, 2022).

Hallnäs on the structure of proofs. As indicated in my autobiographical survey (Schroeder-Heister, 2023, Section 7), I owe to Lars Hallnäs many ideas I consider relevant for current proof-theoretic semantics, and even more relevant for its future development. His idea of definitional reflection, that is, the idea of a general principle to extract information from definitions, which can be partial and are not necessarily monotone, goes way beyond logic and has shaped my understanding of proof-theoretic semantics: not only because this approach represents a powerful extension of logic programming, as we presented it initially (Hallnäs and Schroeder-Heister, 1990), but, more importantly, because it constitutes a general reasoning principle from an intensional point of view. Hallnäs's contribution to this volume sketches the direction into which this might lead when we not only consider the function closure but the functional closure of definitions. Already the structure of natural deduction with its concept of assumption discharge and corresponding side conditions makes such an approach reasonable. The most original idea in this paper is the characterization of proofs in terms of their reductive behaviour, which allows Hallnäs to compare and identify proofs in different formal systems as this behaviour is independent of the formal system itself. I would look at it as an attempt to formalize the concept of 'proof idea', something that every mathematician is aware of and that is the driving force in defining the identity of proofs, but that has not really progressed so far. Hallnäs calls it a "structure theory of proofs" which is much more abstract than "structural proof theory" (Negri and von Plato, 2001) (which partly overlaps with what Prawitz (1971; 1972; 1973) called "general proof theory"). Using his general operators on proofs, he discusses in particular what I have called "Ekman's paradox", a topic of his former doctoral student Jan Ekman to which Hallnäs drew my attention in the early 1990s and which has fascinated me ever since, leading to recent work with Tranchini (Schroeder-Heister and Tranchini 2017; 2021). He can formally demonstrate that both Ekman's normalization paradox and Russell's set-theoretic paradox, though formulated in different formal systems, are based on the same idea, as they satisfy the same abstract proof equation. Hallnäs's analysis actually gives a structural rendering of Ekman's reduction of proof terms, which from the semantical perspective applied by Tranchini and myself remains invisible. The application of such general tools is a promising method in advanced intensional proof-theoretic semantics. This holds likewise for the advanced second-order tools used by Pistone and Tranchini (2023, this volume).

Francez and Kaminski on truth-value constants in multi-valued logics. The contribution by Francez and Kaminski can be seen as an application of proof-theoretic semantics to a system where formulas are signed with truth values. It is thus a generalization of bilateral systems, where one uses positively and negatively signed formulas, to the case of finitely many truth values. The elimination rule of the system formulated in sequent-style natural deduction corresponds to the general elimination rule proposed by Prawitz (1979) and myself (Schroeder-Heister, 1984a), but is now derived from introduction rules based on the truth functional meaning of the connective considered. The paper discusses in particular the case of the nullary constants truth and falsity and their generalizations to arbitrary truth values, and establishes that we have explosion rules for them corresponding to *ex falso quodlibet* in the case when the nullary constant is signed with a non-matching truth value. This shows that proof-theoretic semantics can be productively applied in the area of multi-valued logic and is not confined to intuitionistically inspired logics.

Więckowski on counterfactual assumptions and implications. Więckowski applies proof-theoretic semantics to causal and counterfactual reasoning, more precisely to reasoning from assumptions where assumptions are either *factual* ("since A is the case, B is the case", i.e., "B is the case, because A is the case") or *counterfactual* ("if A were the case, B would be the case"). He overcomes my general characterization of assumptions in natural deduction as "unspecific", which I used to distinguish assumptions in natural deduction from those in bidirectional sequent-based reasoning (Schroeder-Heister, 2004). His idea is to use two proof systems: a "reference system" which is used to infer the assumptions of the "modal system". When the reference system derives the assumption in a canonical way, the modal consequence is a factual or causal inference, while if this is not the case, the modal consequence is counterfactual. Thus the reference system allows one to distinguish between a derivation from an accepted assumption, from a non-accepted assumption, and from an unspecific assumption just laid down. As his reference system he chooses subatomic natural deduction as proposed by himself (Więckowski, 2011), which is particularly suited to deal with identity assumptions in the modal system. This approach is a further step (there are not so many yet[5]) to make proof-theoretic semantics fruitful for the investigation of intensional natural-language phenomena.

Bärtschi and Jäger on set-theoretic reduction principles. The contribution by Bärtschi and Jäger sits on the border between reductive and general proof theory. It investigates the strength of so-called separation principles in second-order arithmetic, which allow one to distinguish two disjoint unary formulas by means of a set containing instances of the first but no instance of the second. These principles play an important role in reverse mathematics. Under the name "reduction principles" (to distinguish them from set-theoretic "separation axioms") they are investigated here with respect to set-theoretic laws, in particular in Kripke-Platek set theory as compared to systems with transfinite recursion. This demonstrates how much remains to be done in proof-theoretic semantics to achieve significant results of mathematical proof theory, given that Kripke-Platek set theory is related to theories of inductive definitions. For me inductive definitions are a key topic in a proof-theoretic semantics with definitional reflection, in particular when functional closure as in Hallnäs's (2023) contribution (this volume) is taken into account.

To conclude, when I look at the breadth and depth of the content of these essays, I feel confirmed in my assessment that proof-theoretic semantics has a bright future.

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[5] One is the second half (Part II) of Francez's (2015) book on proof-theoretic semantics, which deals with applications in linguistics, as well as later work of this author such as Francez (2022).

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