

Outline

- I Theory Brush-Up**
- II Empirical Tests of Portfolio Theory and Individual Investor Behavior**
- III Tests of the CAPM**
- IV Testing the APT**
- V Anomalies or Priced Risk Factors?**
- VI Further Topics**

Applications:

- Explaining the cross-section of expected / past returns
- Estimating expected returns
 - Portfolio management
 - Determining the cost of capital
 - Tests of market efficiency and event studies
 - Benchmark for performance measurement

I. Theory Brush Up

Outline:

1. Portfolio Theory
2. The Capital Asset Pricing Model (CAPM)
3. The Arbitrage Pricing Theory (APT)

I.1. Theory of Portfolio Selection

Objective:

Normative theory that describes how investors should construct their portfolios in order to maximize expected utility

Assumptions:

Frictionless markets:

1. no transaction costs
2. no taxes
3. no indivisibility
4. perfect competition
5. no short sale restrictions

Investor behavior:

6. Investors maximize expected utility at the end of a single period
7. Expected utility only depends on expected end-of-period wealth and variance of end-of-period wealth (μ - σ -valuation)

μ - σ -valuation requires

- quadratic utility (implying decreasing absolute risk aversion)
- normally distributed returns

Return and risk of a portfolio:

Return:

$$E(\tilde{r}_p) \equiv \mu_p = \sum_{i=1}^m x_i \mu_i ; \quad \sum_{i=1}^m x_i = 1$$

Risk:

$$Var(\tilde{r}_p) \equiv \sigma_p^2 = \sum_{i=1}^m \sum_{k=1}^m x_i x_k \sigma_{i,k} = \sum_{i=1}^m x_i^2 \sigma_i^2 + \sum_{i=1}^m \sum_{\substack{k=1 \\ k \neq i}}^m x_i x_k \sigma_{i,k}$$

or alternatively

$$Var(\tilde{r}_p) = \sum_{i=1}^m \sum_{k=1}^m x_i x_k \sigma_i \sigma_k \rho_{i,k} = \sum_{i=1}^m x_i^2 \sigma_i^2 + \sum_{i=1}^m \sum_{\substack{k=1 \\ k \neq i}}^m x_i x_k \sigma_i \sigma_k \rho_{i,k}$$

($\sigma_{i,k}$: covariance; $\rho_{i,k}$: correlation; x_i : portfolio weight)

In matrix notation:

$$\mu_p = \mathbf{x}'\mathbf{r}$$

$$\text{Var}(r_p) = \mathbf{x}'\mathbf{\Omega}\mathbf{x}$$

$$\mathbf{x}'\mathbf{1} = 1$$

$$\text{Cov}(r_q, r_p) = \mathbf{y}'\mathbf{\Omega}\mathbf{x}$$

and

$$\text{Cov}(r_i, r_p) = \mathbf{i}'\mathbf{\Omega}\mathbf{x}$$

where \mathbf{i} is defined such that the i th element is 1 and all other elements are zero

Further:

$$\boldsymbol{\sigma}_{i,P} = \mathbf{\Omega}\mathbf{x}$$

where $\boldsymbol{\sigma}_{i,P}$ is a column vector with elements $\text{Cov}(r_i, r_p)$

Finding the minimum variance portfolio:

Problem:

$$\mathbf{x}'\boldsymbol{\Omega}\mathbf{x} \rightarrow \underset{\mathbf{x}}{\text{Min}}$$

$$s.t. \mathbf{x}'\mathbf{1} = 1$$

Lagrangian:

$$L = \frac{1}{2}\mathbf{x}'\boldsymbol{\Omega}\mathbf{x} + \lambda(1 - \mathbf{x}'\mathbf{1})$$

First order conditions:

$$\frac{\partial L}{\partial \mathbf{x}} = \boldsymbol{\Omega}\mathbf{x} - \lambda\mathbf{1} \stackrel{!}{=} 0 \Rightarrow \boldsymbol{\Omega}\mathbf{x} = \lambda\mathbf{1} \Rightarrow \mathbf{x} = \lambda(\boldsymbol{\Omega}^{-1}\mathbf{1})$$

$$\frac{\partial L}{\partial \lambda} = 1 - \mathbf{x}'\mathbf{1} \stackrel{!}{=} 0 \Rightarrow \mathbf{x}'\mathbf{1} = \mathbf{1}'\mathbf{x} = 1$$

Pre-multiplying the first FOC by $\mathbf{1}'$:

$$\mathbf{1}' \mathbf{x} = 1 = \lambda (\mathbf{1}' \boldsymbol{\Omega}^{-1} \mathbf{1}) \Rightarrow \lambda = \frac{1}{\mathbf{1}' \boldsymbol{\Omega}^{-1} \mathbf{1}}$$

and:

$$\mathbf{x} = \frac{1}{\mathbf{1}' \boldsymbol{\Omega}^{-1} \mathbf{1}} \boldsymbol{\Omega}^{-1} \mathbf{1}$$

Optimal Portfolio Weights:

Problem:

$$\mathbf{x}'\boldsymbol{\Omega}\mathbf{x} \rightarrow \underset{\mathbf{x}}{\text{Min}}$$

$$s.t. \mathbf{x}'\mathbf{r} = \bar{r}$$

$$s.t. \mathbf{x}'\mathbf{1} = 1$$

Lagrangian:

$$L = \frac{1}{2}\mathbf{x}'\boldsymbol{\Omega}\mathbf{x} + \lambda_1(1 - \mathbf{x}'\mathbf{1}) + \lambda_2(\bar{r} - \mathbf{x}'\mathbf{r})$$

First order conditions:

$$\frac{\partial L}{\partial \mathbf{x}} = \boldsymbol{\Omega}\mathbf{x} - \lambda_1\mathbf{1} - \lambda_2\mathbf{r} \stackrel{!}{=} 0 \Rightarrow \boldsymbol{\Omega}\mathbf{x} = \lambda_1\mathbf{1} + \lambda_2\mathbf{r} \Rightarrow \mathbf{x} = \lambda_1(\boldsymbol{\Omega}^{-1}\mathbf{1}) + \lambda_2(\boldsymbol{\Omega}^{-1}\mathbf{r})$$

$$\frac{\partial L}{\partial \lambda_1} = 1 - \mathbf{x}'\mathbf{1} \stackrel{!}{=} 0 \quad ; \quad \frac{\partial L}{\partial \lambda_2} = \bar{r} - \mathbf{x}'\mathbf{r} \stackrel{!}{=} 0$$

Pre-multiplying the first FOC by $\mathbf{1}'$:

$$\mathbf{1}' \mathbf{x} = 1 = \lambda_1 (\mathbf{1}' \boldsymbol{\Omega}^{-1} \mathbf{1}) + \lambda_2 (\mathbf{1}' \boldsymbol{\Omega}^{-1} \mathbf{r})$$

Pre-multiplying the first FOC by \mathbf{r}' :

$$\mathbf{r}' \mathbf{x} = \bar{r} = \lambda_1 (\mathbf{r}' \boldsymbol{\Omega}^{-1} \mathbf{1}) + \lambda_2 (\mathbf{r}' \boldsymbol{\Omega}^{-1} \mathbf{r})$$

Solving for λ_1 and λ_2 and reinserting into the FOV yields the solution

Solving for alternative values of \bar{r} yields the efficient frontier

I.2. The Capital Asset Pricing Model (CAPM)

Characterization:

- Theory of portfolio selection is embedded in a capital market equilibrium

Assumptions:

Frictionless markets:

1. no transaction costs
2. no taxes
3. no indivisibility
4. perfect competition
5. no short sale restrictions
6. riskless lending and borrowing at the rate r_f

7. all assets are tradeable

Investor behavior:

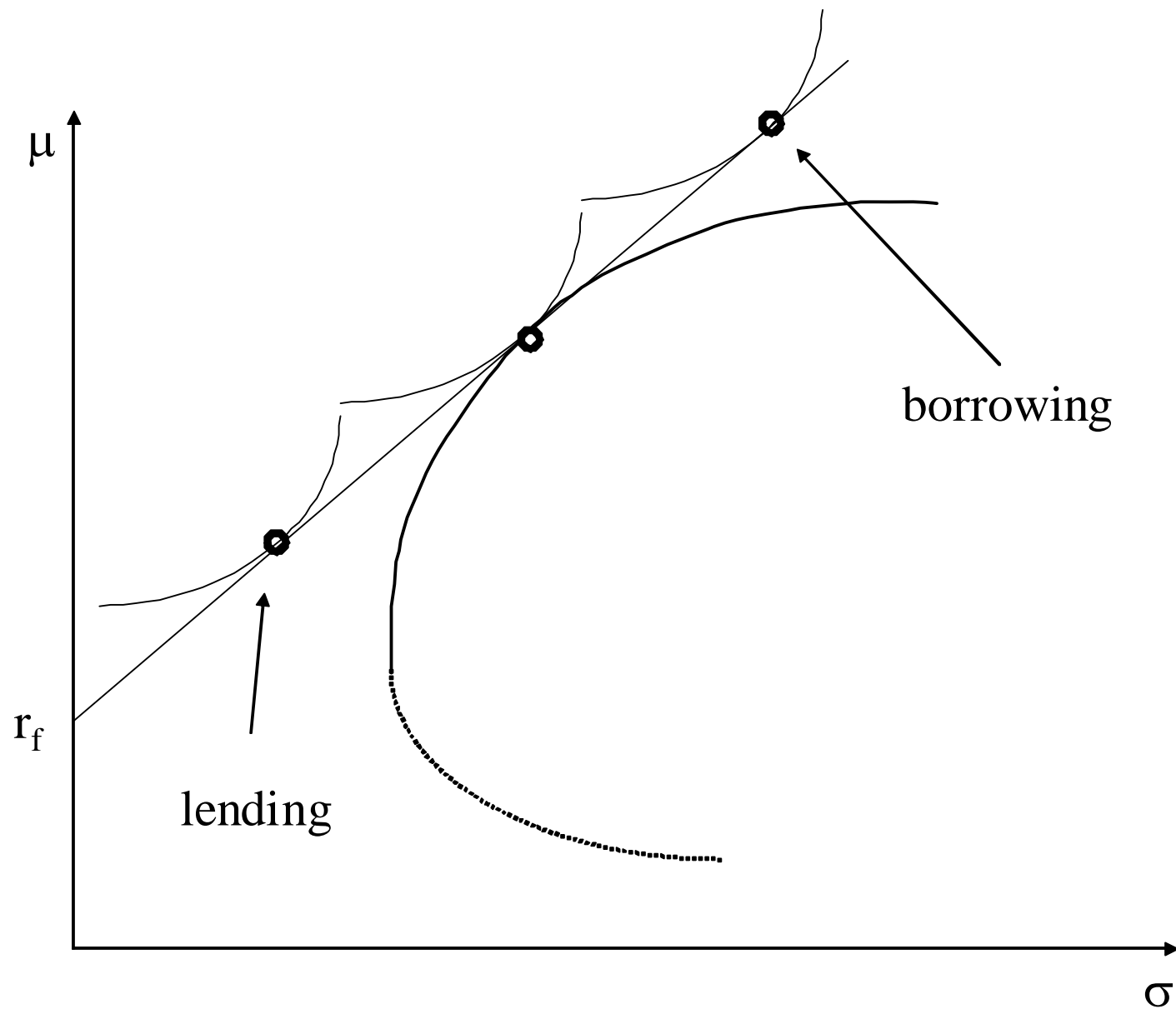
8. μ - σ -valuation

9. homogeneous expectations

10. identical time horizon

It follows

- All investors calculate the same efficient frontier
- Two-fund separation



The market portfolio:

- All investors hold portfolios of risky assets with identical weights
- Equilibrium requires that these weights be identical to

$$x_i = \frac{n_i p_i}{\sum_{j=1}^m n_j p_j} > 0$$

Expected return and variance of the return on the market portfolio

$$E(\tilde{r}_m) = \mu_m = \sum_{i=1}^m x_i \mu_i; \quad \sum_{i=1}^m x_i = 1$$

$$\text{Var}(\tilde{r}_m) = \sum_{i=1}^m \sum_{k=1}^m x_i x_k \sigma_{i,k} = \sum_{i=1}^m x_i \sum_{k=1}^m x_k \sigma_{i,k} = \sum_{i=1}^m x_i \text{Cov}(\tilde{r}_i, \tilde{r}_m) = \sum_{i=1}^m x_i \sigma_{i,m}$$

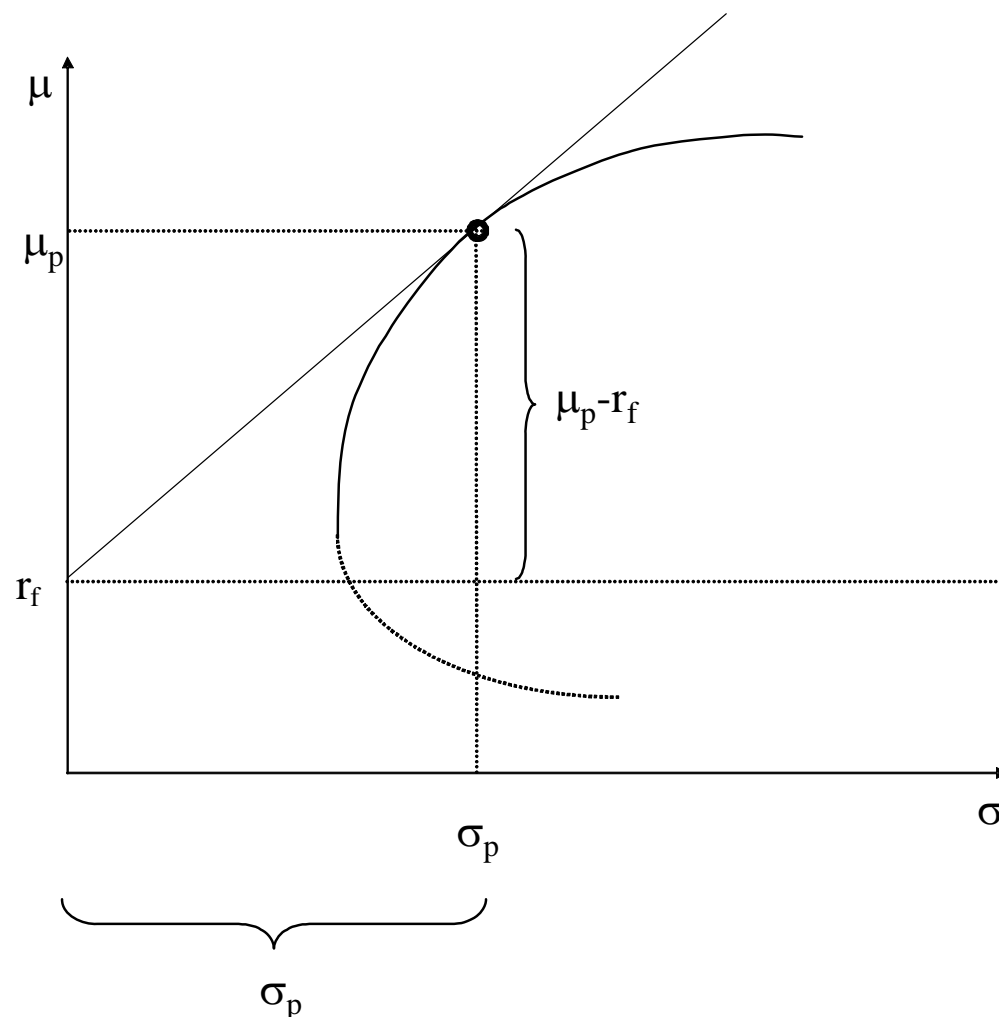
$\text{Var}(\tilde{r}_m)$ = weighted average covariance of the return of each risky security with the market portfolio

This is the non-diversifiable (=systematic) risk. An individual stock's contribution to this risk is

$$\frac{\partial \sigma_m}{\partial x_i} = \frac{1}{2\sigma_m} \left(2 \sum x_k \sigma_{i,k} \right) = \frac{\sigma_{i,m}}{\sigma_m} = \beta_i \sigma_m ; \beta_i = \frac{\sigma_{i,m}}{\sigma_m^2}$$

„...through diversification, some of the risk inherent in an asset can be avoided so that its total risk is obviously not the relevant influence on its price...“ (Sharpe 1964)

Deriving the CAPM



The optimal portfolio has the maximal slope (= the maximal **Sharpe Ratio**)

$$\theta = \frac{r_p - r_f}{\sigma_p} = \frac{\mathbf{x}'\mathbf{r} - r_f}{\sqrt{\mathbf{x}'\boldsymbol{\Omega}\mathbf{x}}}$$

Maximization problem:

$$\theta = \frac{\mathbf{x}'\mathbf{r} - r_f}{\sqrt{\mathbf{x}'\boldsymbol{\Omega}\mathbf{x}}} \rightarrow \underset{\mathbf{x}}{Max}$$

$$s.t. \mathbf{x}'\mathbf{1} = 1$$

Since $\mathbf{x}'\mathbf{1}r_f = \mathbf{x}'\mathbf{r}_f = r_f$

$$\theta = \frac{\mathbf{x}'\mathbf{r} - \mathbf{x}'\mathbf{r}_f}{\sqrt{\mathbf{x}'\boldsymbol{\Omega}\mathbf{x}}} = \frac{\mathbf{x}'(\mathbf{r} - \mathbf{r}_f)}{\sqrt{\mathbf{x}'\boldsymbol{\Omega}\mathbf{x}}} \rightarrow \underset{\mathbf{x}}{Max}$$

FOC:

$$\frac{\partial \theta}{\partial \mathbf{x}} = \frac{(\mathbf{r} - \mathbf{r}_f) \sqrt{\mathbf{x}' \boldsymbol{\Omega} \mathbf{x}} - \mathbf{x}' (\mathbf{r} - \mathbf{r}_f) \frac{1}{2\sqrt{\mathbf{x}' \boldsymbol{\Omega} \mathbf{x}}} 2\boldsymbol{\Omega} \mathbf{x}}{\mathbf{x}' \boldsymbol{\Omega} \mathbf{x}} = 0$$

which yields

$$(\mathbf{r} - \mathbf{r}_f) = \frac{\mathbf{x}' (\mathbf{r} - \mathbf{r}_f) \boldsymbol{\Omega} \mathbf{x}}{\mathbf{x}' \boldsymbol{\Omega} \mathbf{x}}$$

Since $\boldsymbol{\Omega} \mathbf{x} = \boldsymbol{\sigma}_{i,P}$ and $\text{Var}(r_p) \equiv \sigma_p^2 = \mathbf{x}' \boldsymbol{\Omega} \mathbf{x}$ and $\mathbf{x}' (\mathbf{r} - \mathbf{r}_f) = (r_p - r_f)$

$$\mathbf{r} = \mathbf{r}_f + (r_p - r_f) \frac{\boldsymbol{\sigma}_{i,P}}{\sigma_p^2}$$

- There is a linear relation between the return on individual securities and the return on the tangency portfolio

- Market clearing requires that the tangency portfolio be the market portfolio, i.e.

$$\mathbf{x} = \frac{1}{\mathbf{n} \mathbf{p}} \mathbf{n} \mathbf{1}_{m \times m} \mathbf{p}$$

where \mathbf{n} is a vector of the number of shares and \mathbf{p} is the price vector; m denotes the number of risky assets

- Therefore

$$\mathbf{r} = \mathbf{r}_f + \boldsymbol{\beta} (r_m - r_f); \boldsymbol{\beta} = \frac{1}{\sigma_m^2} \boldsymbol{\sigma}_{i,m}$$

the CAPM

Testable implications:

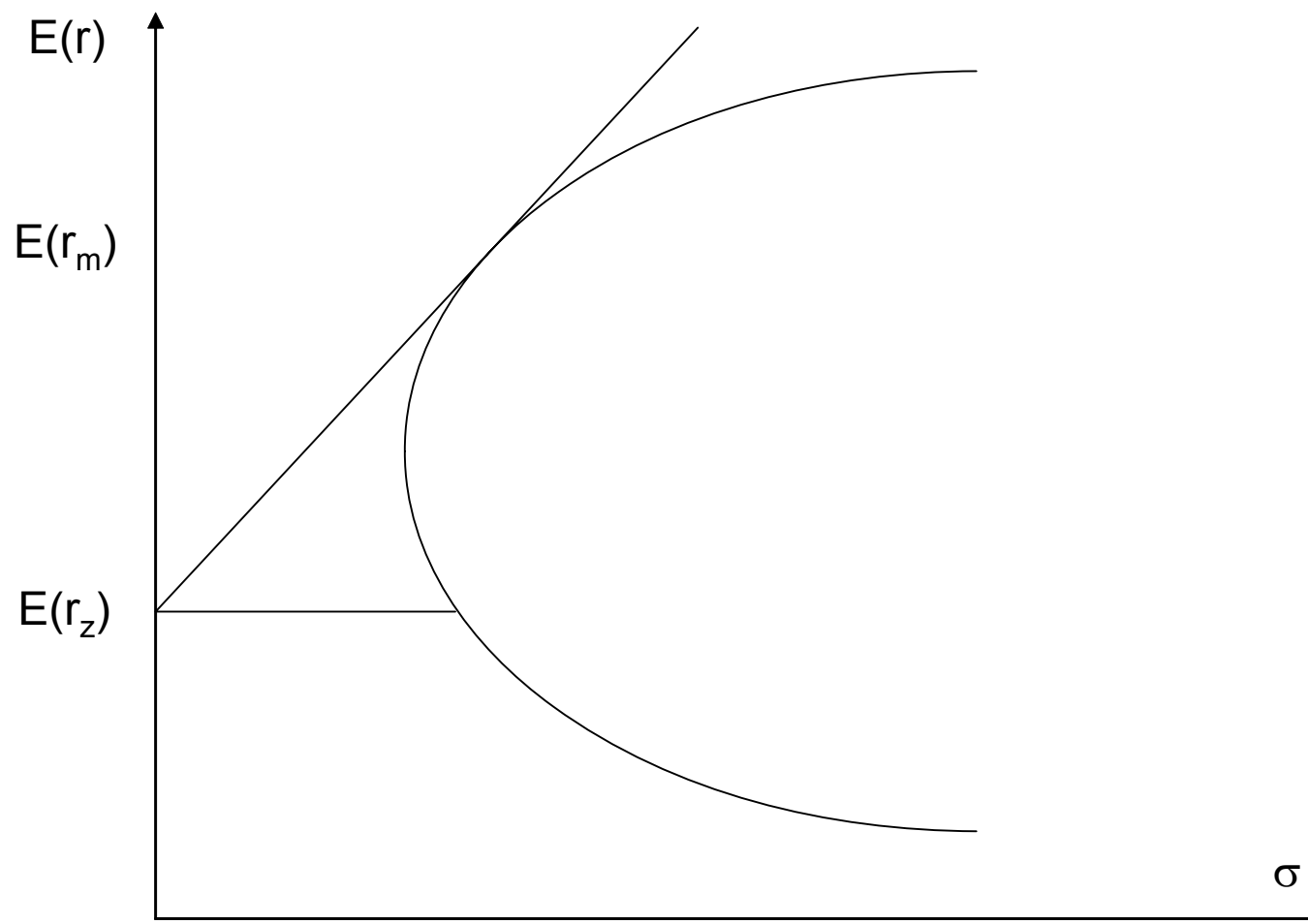
1. In a regression of the risk premia of stocks / portfolios on the market risk premium, the intercept α_i should be zero

$$(r_{i,t} - r_{f,t}) = \alpha_i + \beta_i (r_{m,t} - r_{f,t})$$

2. The market risk premium is positive
3. Beta is the only determinant of the cross-sectional variation of expected risk premia

No Riskless Asset

The Zero-Beta CAPM (Black):



$$E(r_i) = E(r_z) + \beta_i [E(r_m) - E(r_z)]$$

Testable implication:

- $E(r_i) = (1 - \beta_i)E(r_z) + \beta_i E(r_m)$
- $E(r_z)$ is unobservable
- In a time-series regression for several stocks / portfolios, the intercept vector should satisfy

$$\alpha = (\mathbf{1} - \beta) \lambda$$

- This restriction can be tested (Gibbons 1982)

I.3. The Arbitrage Pricing Theory (APT)

Basic idea (Ross 1976):

- Starting point: „return generating process“ is a linear factor model
- No arbitrage in equilibrium
Arbitrage: zero investment, no risk, positive future payoff with $p > 0$
- Under some additional assumptions, a linear expression for the expected returns can be derived
- No assumptions on preferences necessary (in the basic APT)

Result: The expected return on an asset is a linear function of the risk premia on a (fixed) number of (economically meaningful) factors

Sketch of the derivation

Assumptions:

1. Realized returns are generated by a linear factor model („return generating process“):

$$r_i = a_i + b_{i,1}X_1 + b_{i,2}X_2 + \dots + b_{i,k}X_k + \varepsilon_i$$

or

$$r_i = E(r_i) + b_{i,1}\delta_1 + b_{i,2}\delta_2 + \dots + b_{i,k}\delta_k + \varepsilon_i$$

where

$$\delta_j = X_j - E(X_j)$$

such that

$$E(\delta_j) = 0 \quad \forall j$$

Assumptions on the error term:

$$E(\varepsilon_i) = 0 \quad \forall i$$

$$E(\varepsilon_i \delta_j) = 0 \quad \forall i, j$$

$$E(\varepsilon_i \varepsilon_k) = 0 \quad \forall i, k \text{ mit } i \neq k$$

Note: $E(\varepsilon_i \varepsilon_k) = 0$ is assumed in the (basic) APT but not in the CAPM(!) (Ross / Westerfield / Jaffe S. 305)

2. No arbitrage:

- Frictionless market; no taxes, no transaction costs, no short-sale restrictions
- A large number of assets
- Homogenous expectations

We construct a portfolio that requires no investment (short sales!) and has no systematic risk, i.e.:

$$\sum_{i=1}^m x_i p_i = 0 \quad \text{and} \quad \sum_{i=1}^m x_i b_{i,j} = 0 \quad \forall j$$

x_i : weight of asset i in the portfolio

The unsystematic risk is eliminated through diversification; i.e.:

$$\sum_{i=1}^m x_i \varepsilon_i \approx 0, \quad \text{und} \quad \text{Var} \left(\sum_{i=1}^m x_i \varepsilon_i \right) \approx 0$$

[This requires, strictly speaking, an infinite economy, i.e. $m \rightarrow \infty$]

The return on this portfolio must be 0 (no arbitrage). From this, a linear expression for the expected return can be derived (Ross 1976)

$$E(r_i) \equiv \mu_i \approx \lambda_0 + b_{i,1}\lambda_1 + b_{i,2}\lambda_2 + \dots b_{i,k}\lambda_k$$

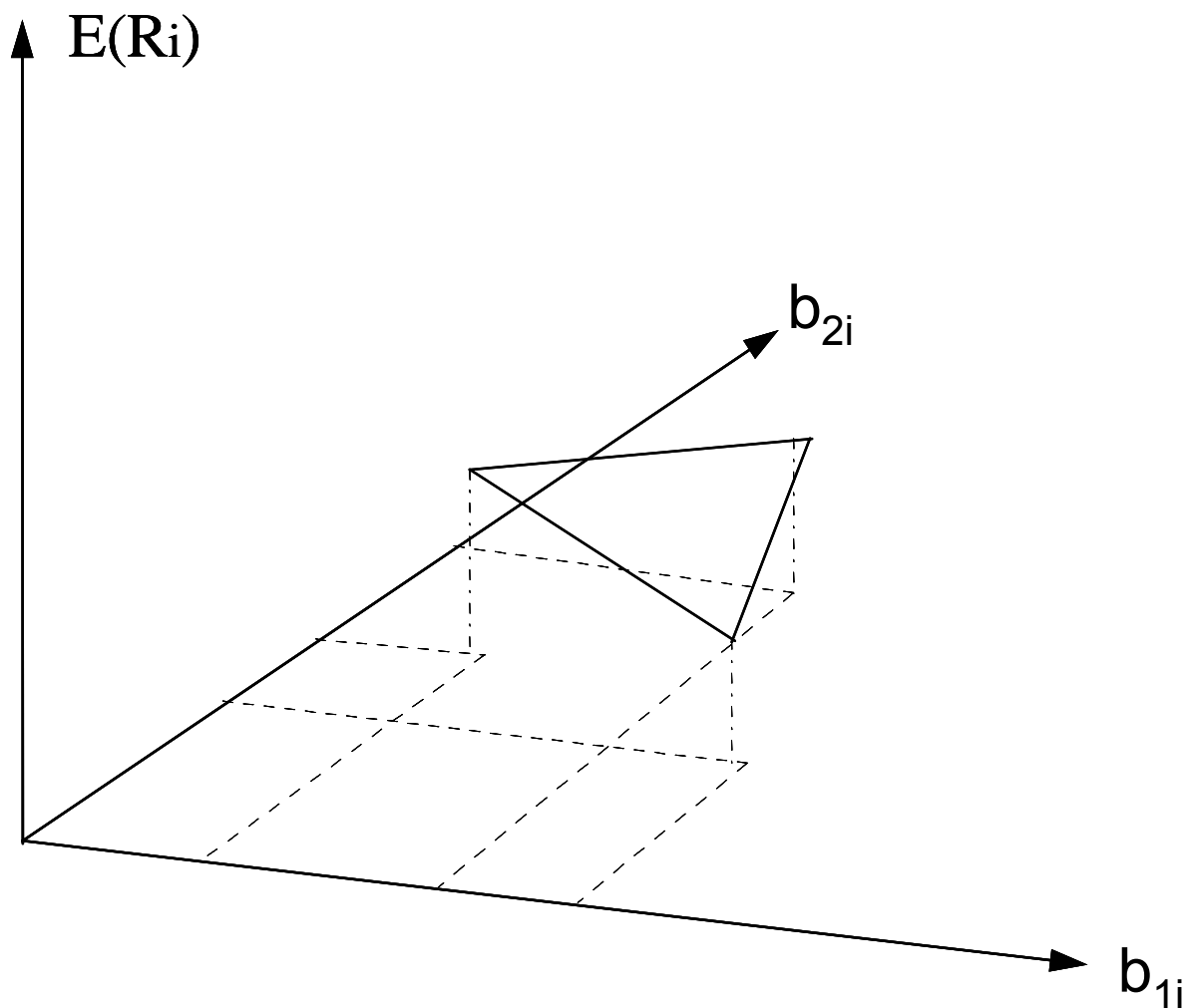
This relation is only an approximation in a finite economy (though there are conditions under which exact factor pricing obtains).

The λ_j are risk premia, the $b_{i,j}$ are sensitivities (or factor loadings)

If riskless lending is possible, λ_0 is the riskless return

An important point: Since the no-arbitrage condition must hold for any subset of assets, identification of the market portfolio is not necessary

Graphical representation for $k = 2$:



Assume a two-factor APT: $\mu_i = \lambda_0 + b_{i1} \lambda_1 + b_{i2} \lambda_2$

The return on an asset with no systematic risk is :

$$\mu_Z = \lambda_0 + 0 \cdot \lambda_1 + 0 \cdot \lambda_2 \Leftrightarrow \lambda_0 = \mu_Z (= r_f)$$

An asset with $b_{1,1} = 1$ and $b_{1,2} = 0$ has the expected return

$$\mu_1 = \mu_Z + 1 \cdot \lambda_1 + 0 \cdot \lambda_2 \Leftrightarrow \lambda_1 = (\mu_1 - \mu_Z)$$

and similarly:

$$\mu_2 = \mu_Z + 0 \cdot \lambda_1 + 1 \cdot \lambda_2 \Leftrightarrow \lambda_2 = (\mu_2 - \mu_Z)$$

Therefore:

$$\mu_i = \mu_Z + b_{i1} (\mu_1 - \mu_Z) + b_{i2} (\mu_2 - \mu_Z)$$

The portfolios characterized by μ_1 , μ_2 are **mimetic** (factor-substituting) portfolios or **basis portfolios** (Lehman and Modest 1988).

Testability:

"In its most general form the APT provides an *approximate* relation for asset returns with an unknown number of unidentified factors. At this level rejection of the theory is impossible (unless arbitrage opportunities exist) and as a consequence testability of the model depends on the introduction of additional assumptions."

(Campbell et al. 1997, p. 220)

Note: The intertemporal CAPM (ICAPM, Merton 1973) also delivers a multifactor model

CAPM and APT

- APT requires less stringent assumptions (no $\mu - \sigma$ - preferences, no identification of the market portfolio required)
(note though: derivation of the APT in a finite economy does require assumptions on investor preferences; "equilibrium APT")
- APT does not identify the factors
- A one-factor APT with the market return being the single factor is formally identical to the CAPM (cf Ross / Westerfield / Jaffe 1996, S. 304)
- A multi-factor APT is not necessarily incompatible with the CAPM

Let

$$r_i = a_i + b_{i,1}I_1 + b_{i,2}I_2 + \varepsilon_i$$

be the return generating process and let

$$\mu_i = r_f + b_{i,1}\lambda_1 + b_{i,2}\lambda_2$$

be the expected return equation (2-factor APT)

Rewrite:

$$\mu_i = r_f + b_{i1}(\mu_1 - r_f) + b_{i2}(\mu_2 - r_f)$$

Let β_{λ_1} and β_{λ_2} be the CAPM betas of the factor-substituting portfolios. Then

$$\mu_1 = r_f + \beta_{\lambda_1}(\mu_m - r_f); \mu_2 = r_f + \beta_{\lambda_2}(\mu_m - r_f).$$

Inserting into the APT yields :

$$\begin{aligned} \mu_i &= r_f + b_{i1}[\beta_{\lambda_1}(\mu_m - r_f)] + b_{i2}[\beta_{\lambda_2}(\mu_m - r_f)] \\ &= r_f + [b_{i1}\beta_{\lambda_1} + b_{i2}\beta_{\lambda_2}](\mu_m - r_f) \\ &= r_f + \beta_i^*(\mu_m - r_f); \beta_i^* = [b_{i1}\beta_{\lambda_1} + b_{i2}\beta_{\lambda_2}] \end{aligned}$$

APT and CAPM are compatible.

Background:

- The CAPM does not require $Cov(\varepsilon_i; \varepsilon_j) = 0 \forall i, j; i \neq j$. Therefore, the CAPM does not assume that the return on the market portfolio is the sole source of covariation between asset returns.
- APT models those alternative sources of covariation

II. Empirical Tests of Portfolio Theory and Individual Investor Behavior

Outline:

1. Measuring Risk and Return
2. The Value of Diversification
3. Individual Investors' Portfolios
4. Do Individual Errors Cancel Out?

Testable predictions:

1. Investors hold well-diversified portfolios
2. All investors hold identical portfolios of risky assets

II.1. Measuring Risk and Return

Measuring Return

- Starting point: discrete holding period return

$$r_t = \frac{P_1 - P_0 + D}{P_0}$$

Average return of one asset over several periods:

- Arithmetic mean?
- Problem 1: Intermediate cash inflows and cash outflows require weighting of single period returns
- Problem 2: Changes in value yield different start-of-period wealth even without cash in- and outflows
- Therefore: several "mean concepts" are needed

Money-weighted return:

- The money-weighted return equals the internal rate of return of the cash flow stream

$$\sum_{t=1}^T \frac{C_t}{(1 + r_{mw})^t} = 0$$

- It measures the average change in value of a portfolio while allowing cash in- and outflows

Example: Regular investment in a fund over three years

Year	Investment on Jan 1st	Return
1	1000	+20,0%
2	1000	-10,00%
3	1000	+5,0%

Wealth at end of year 3:

$$1000(1+0,2)(1-0,1)(1+0,05) + 1000(1-0,1)(1+0,05) + 1000(1+0,05) = 3129$$

Money-weighted return:

$$-1000 - \frac{1000}{(1+r_{mw})} - \frac{1000}{(1+r_{mw})^2} + \frac{3129}{(1+r_{mw})^3} = 0 \Leftrightarrow r_{mw} \approx 2,12\%$$

Is the money-weighted return a useful measure of the performance of the fund manager?

Time-weighted return:

- The time-weighted return is the geometric average of the growth factors in the individual periods:

$$\bar{r} = \left[\prod_{t=1}^T (1 + r_t) \right]^{\frac{1}{T}} - 1$$

- In our example: $[1,2 \cdot 0,9 \cdot 1,05]^{\frac{1}{3}} - 1 = 0,0428 \approx 4,28\%$
- Time- and money-weighted returns are equal if there are no intermediate cash in- or outflows:

$$-V_0 + \frac{V_T}{(1 + r_{mw})^T} = 0 \Rightarrow r_{mw} = \left[\frac{V_T}{V_0} \right]^{\frac{1}{T}} - 1 = \left[\frac{V_0 \prod_{t=1}^T (1 + r_t)}{V_0} \right]^{\frac{1}{T}} - 1 = \bar{r}$$

What about the arithmetic mean ?

Example:

- Arithmetic mean of annual DAFOX returns 1967 - 2004: 10,81%
- $1000 \cdot 1,1081^{37} = 44.542,79$
- True terminal value: 19.975,02
- Geometric mean return: 8,430%
- $1000 \cdot 1,0843^{37} = 19.975,02$

Thus:

- The geometric mean is the correct measure for the increase in wealth of a buy-and-hold investment
- The geometric mean is smaller than the arithmetic mean (exception: individual period returns are identical)

- Using the arithmetic mean is appropriate if cash in- and outflows are determined such that the amount invested is identical at the beginning of each period

Estimating returns from historical data:

- Assume you want to estimate the expected return on the DA-FOX for 2005 from the 1967-2004 data
- Assume further that the return distribution is stable
- Then the arithmetic mean provides an unbiased estimate of the next period return

Intra-period returns

- Let r be the return (on a p.a. basis), and n the number of sub-periods (e.g. 12 months).
- Then:

$$V_1 = V_0 \left(1 + \frac{r}{n} \right)^n$$

- For $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} \left[V_0 \left(1 + \frac{r}{n} \right)^n \right] = V_0 e^r$$

→ **continuously compounded return**

Proof:

$$\lim_{n \rightarrow \infty} \left[1 + \frac{r}{n} \right]^n = \lim_{n \rightarrow \infty} \left[1 + \frac{1}{\left(\frac{n}{r} \right)} \right]^{\left(\frac{n}{r} \right)^r}$$

Let $x \equiv \frac{n}{r}$:

$$\lim_{x \rightarrow \infty} \left[\left(1 + \frac{1}{x} \right)^x \right]^r = e^r$$

because by definition

$$e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x$$

Discrete versus continuously compounded returns:

- Terminal wealth after t periods with continuous compounding

$$V_t = V_0 e^{rt}$$

- Let r_d be the discrete return yielding the same terminal wealth:

$$V_0 (1 + r_d) = V_0 e^r \Rightarrow r_d = e^r - 1$$

- and similarly:

$$r = \ln(1 + r_d)$$

Logarithmic returns:

- Log returns are often used in empirical research :

$$r = \ln(P_1) - \ln(P_0) = \ln\left(\frac{P_1}{P_0}\right)$$

- Log returns are equivalent to continuously compounded returns:

$$P_1 = P_0 e^r \Rightarrow \ln\left(\frac{P_1}{P_0}\right) = \ln(e^r) = r$$

- Why log returns?
 - Log returns better fit the normality assumption
 - Log returns are additive, i.e., $r_T = \sum_{t=1}^T r_t$ because

$$\begin{aligned}\sum_{t=1}^T r_t &= \sum_{t=1}^T [\ln(P_t) - \ln(P_{t-1})] \\ &= \ln(P_1) - \ln(P_0) + \ln(P_2) - \ln(P_1) + \dots + \ln(P_T) - \ln(P_{T-1}) \\ &= \ln(P_T) - \ln(P_0) = r_T\end{aligned}$$

- Disadvantage: no portfolio characteristic , i.e.

$$\frac{\ln\left(\sum_{i=1}^n x_i P_{i,1}\right)}{\ln\left(\sum_{i=1}^n x_i P_{i,0}\right)} \neq \sum_{i=1}^n x_i \ln\left(\frac{P_{i,1}}{P_{i,0}}\right)$$

Estimating and Measuring Volatility

- Measure of risk: variance / standard deviations of returns
- Alternative risk measures do exist (e.g. LPM measures)

Moving average

- Estimating standard deviation from T periods of historical data:

$$\hat{\sigma}_T = \sqrt{\frac{1}{T-1} \sum_{t=1}^T (r_t - \bar{r}_T)^2}$$

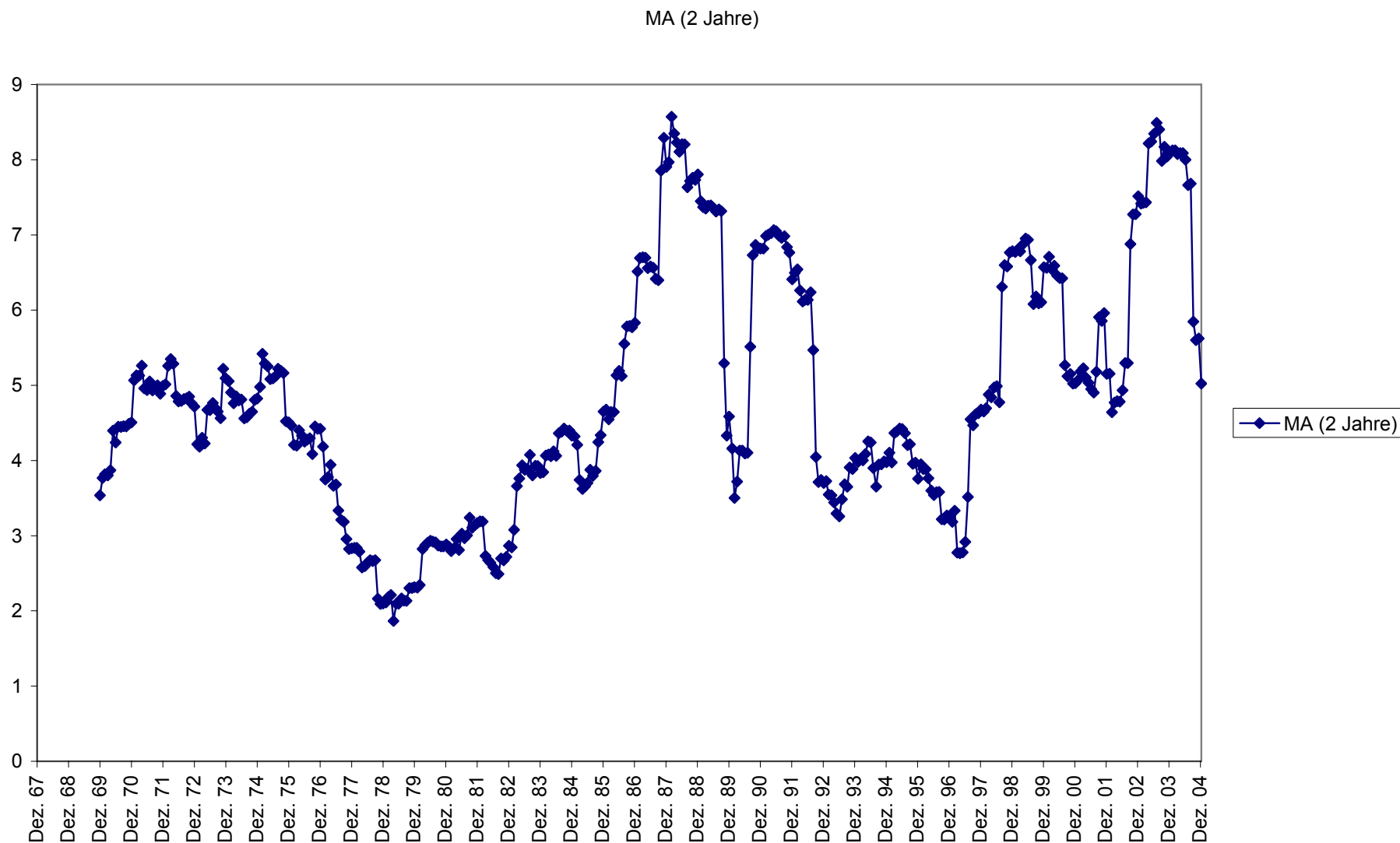
- Alternatively (particularly with daily data) assuming a zero mean return (because the mean is estimated imprecisely)

$$\hat{\sigma}_T = \sqrt{\frac{1}{T} \sum_{t=1}^T (r_t)^2}$$

How much data to use?

- more data yields more precise estimates
- but increased probability of structural breaks

Example: 2-year moving average from monthly DAFOX returns



Characteristics:

- Extreme values affect estimation for exactly the period over which the average is calculated (plateau effect)
- All observations receive equal weight, the weight drops to zero after T periods
- The ordering of the observations is irrelevant
→ potential patterns can not be taken into account

Exponentially weighted moving average:

- More recent observations receive higher weight
- Weight of most recent observation: $(1 - \lambda) \lambda^0$
- Weight of second observation: $(1 - \lambda) \lambda^1$
- Weight of observation T: $(1 - \lambda) \lambda^{T-1}$

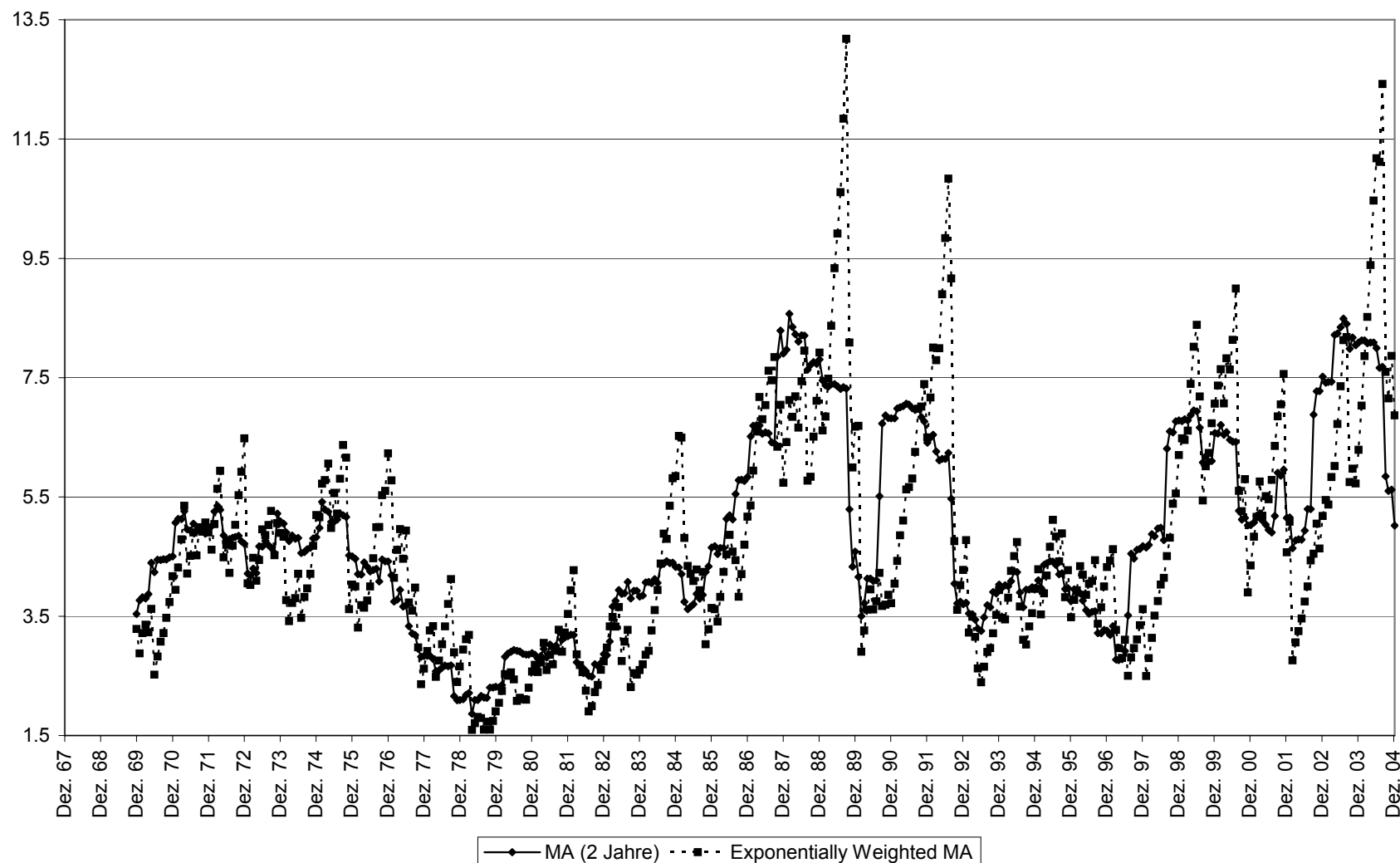
- Parameter λ ($0 < \lambda < 1$) determines the relative weight of "old" and recent observations
- For small T standardization is necessary:

$$\hat{\sigma}_T = \sqrt{\frac{\sum_{t=1}^T (1-\lambda)\lambda^{t-1} (r_t)^2}{\sum_{t=1}^T (1-\lambda)\lambda^{t-1}}}$$

Characterization:

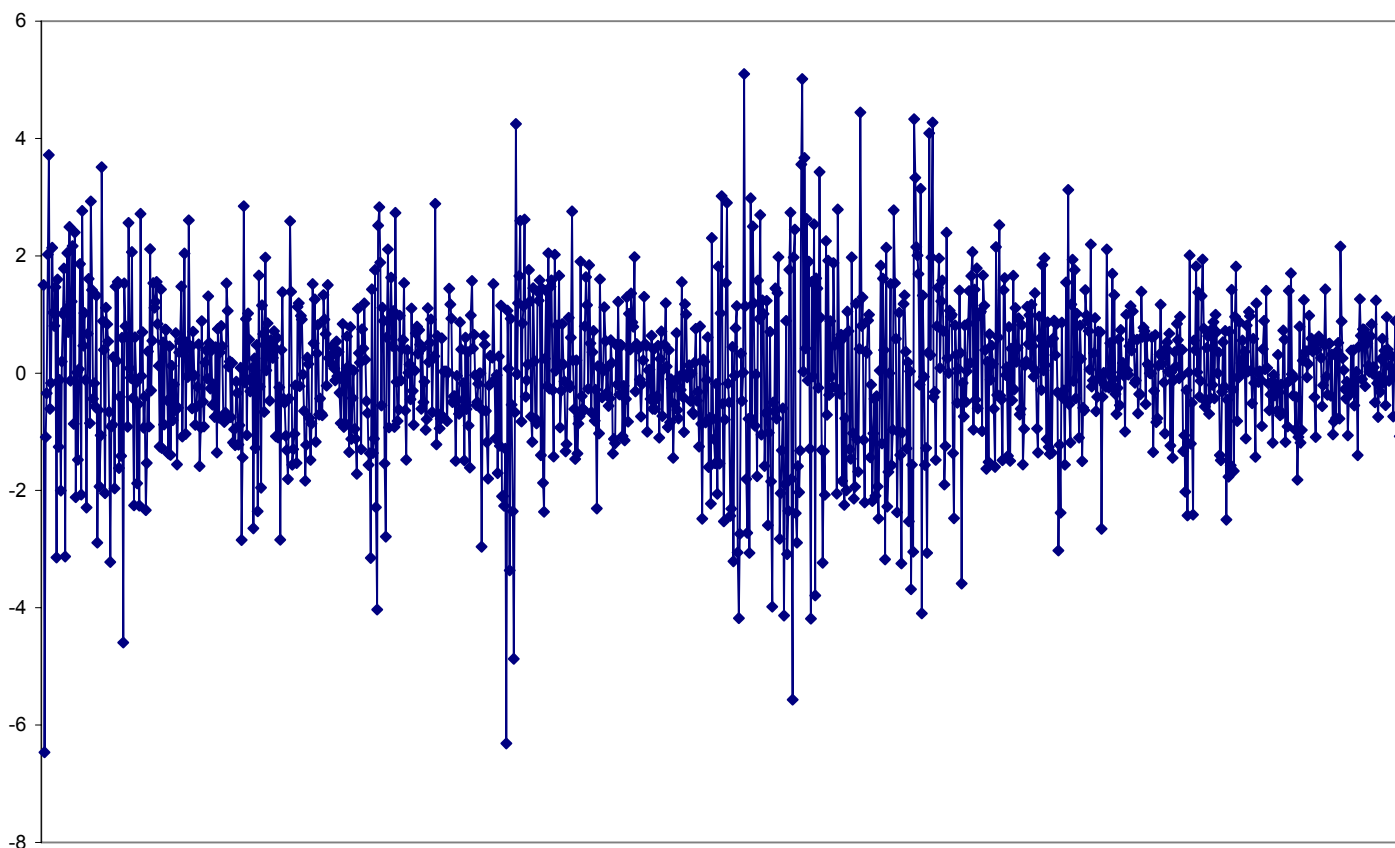
- Avoids the abrupt drop of the weight after T periods
- Recent observations receive very high weight
e.g.: if $\lambda = 0,8$ the most recent observations receives a weight of 20%, the seven most recent observations 80%
- Does an extreme observation predict a permanent increase in volatility?

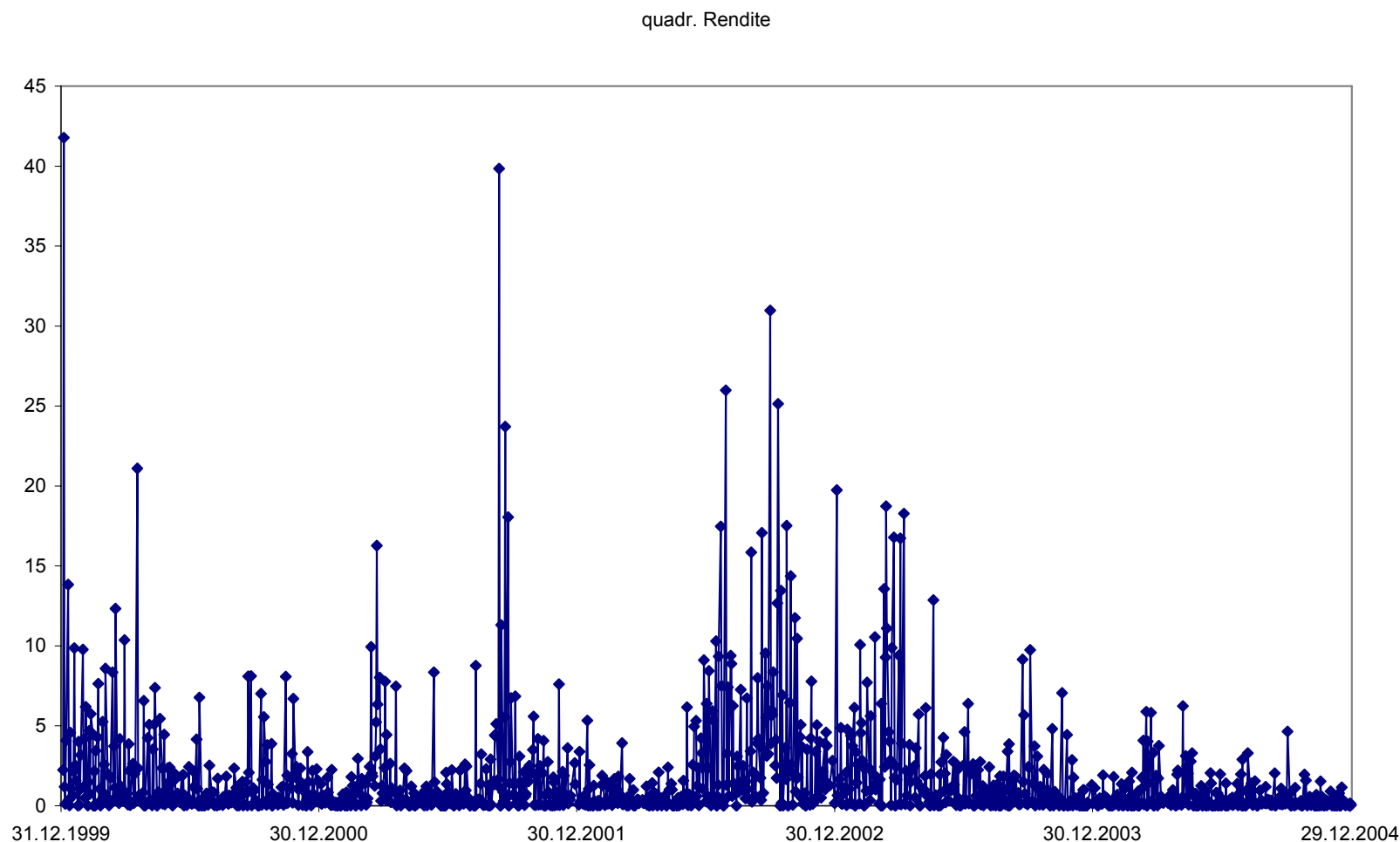
Moving average versus exponentially weighted moving average:



Volatility Clustering

- Consider daily returns and squared daily returns (two years of data):





- we observe volatility clustering: squared returns (and, by implication, volatility) depend on lagged squared returns

- Squared returns are predictable
- This can be modelled using the **(G)ARCH** [(generalized) autoregressive conditional heteroscedasticity] framework
- A GARCH(1,1):

$$r_t = c + \varepsilon_t$$
$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

In our example:

Dependent Variable: R_DAFOX

Method: ML - ARCH (Marquardt) - Normal distribution

Date: 08/05/05 Time: 16:05

Sample (adjusted): 1 1269

Included observations: 1269 after adjustments

Convergence achieved after 10 iterations

Variance backcast: ON

GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.020937	0.030044	0.696867	0.4859
Variance Equation				
C	0.020456	0.008008	2.554482	0.0106
RESID(-1)^2	0.089772	0.012575	7.138745	0
GARCH(-1)	0.897119	0.014367	62.44407	0
R-squared	-0.000853	Mean dependent var		-0.017551
Adjusted R-squared	-0.003226	S.D. dependent var		1.318607
S.E. of regression	1.320732	Akaike info criterion		3.159269
Sum squared resid	2206.582	Schwarz criterion		3.17549
Log likelihood	-2000.556	Durbin-Watson stat		1.912436

- Predicting volatility using GARCH models: Calculate $E\left(\sigma_{t+\tau}^2\right)$ from the model

Topics not covered:

- implied volatility
- realized volatility

II.2. The Value of Diversification

What is a "well-diversified" portfolio?

- Traditionally a portfolio of about 10 stocks has been considered to be well diversified (e.g. Evans and Archer 1968)
- Statman (1987):
 - randomly draw m portfolios of n S&P500 stocks
 - assume their expected return equals the return on the S&P500
 - estimate the expected standard deviation by averaging over the m portfolios
 - construct a portfolio consisting of the S&P 500 and riskless lending/borrowing that has the same standard deviation
 - compare the returns on the two portfolios

- result: 30 [40] stocks for a lending [borrowing] investor are required for a well-diversified portfolio (different lending and borrowing rates are used)
- Campbell et al. (2001):
 - use an extensive data set covering 1962-1997
 - find that firm-specific (idiosyncratic) volatility has increased over time
 - the number of stocks needed to achieve a given level of diversification has increased

The value of diversifying internationally:

- Jorion (2003): monthly returns 1921-1996 from 30 countries

horizon	return (annualized)			volatility (annualized)			Probability of loss		
	1 y.	5 y.	10 y.	1 y.	5 y.	10 y.	1 y.	5 y.	10 y.
lowest-risk country	3,17	0,96	0,38	18,8	7,87	5,00	42,2	43,2	47,6
highest-risk country	6,30	-2,65	-4,33	70,4	19,9	13,7	60,0	61,3	68,7
mean	4,65	1,23	0,38	30,3	12,0	7,78	48,6	46,8	49,9
median	4,45	1,39	0,83	23,1	10,3	6,67	48,2	46,8	48,2
global portfolio	5.11	3.73	3,02	16,1	8,33	5,65	37,8	35,4	35,2

- International diversification is clearly beneficial

Naive versus optimal diversification

- Naive diversification: Invest fraction $1/n$ in each of n assets
- Necessarily inferior to optimal diversification ex post
- Not necessarily inferior ex ante

Problems of optimal portfolio selection:

- Portfolio weights are sensitive to parameter estimates (expected returns in particular)
- Procedure tends to recommend extreme portfolios
- Estimation error is important

An instructive example (deMiguel et al. 2006):

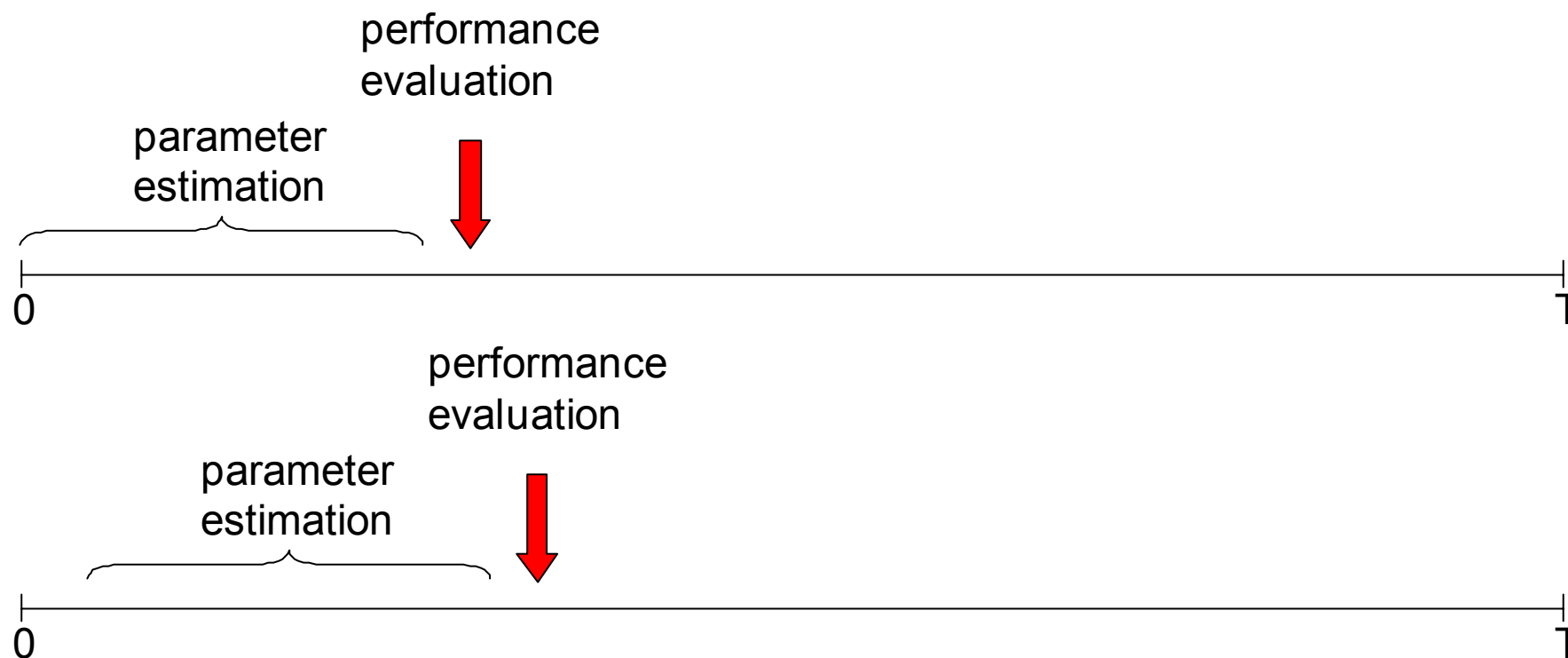
- 2 identical assets with $\mu_i = 0.08$, $\sigma_i = 0.20$, $\rho = 0.99$
- optimal weights: 0.5
- assume expected return on asset 1 is erroneously estimated to be 0.09
- "optimal" weights then: $x_1 = 6.35$, $x_2 = -5.35$

Portfolio optimization models that take estimation error into account:

- Bayesian approaches (diffuse priors, shrinkage estimators, model-based priors)
- Moment restrictions (e.g. the minimum variance portfolio)
- Portfolios with short-sale constraints (avoids extreme portfolios)
- Combinations of portfolios

The study of deMiguel et al. (2006):

- "Horse race" of 14 models
- Rolling-window out of sample analysis

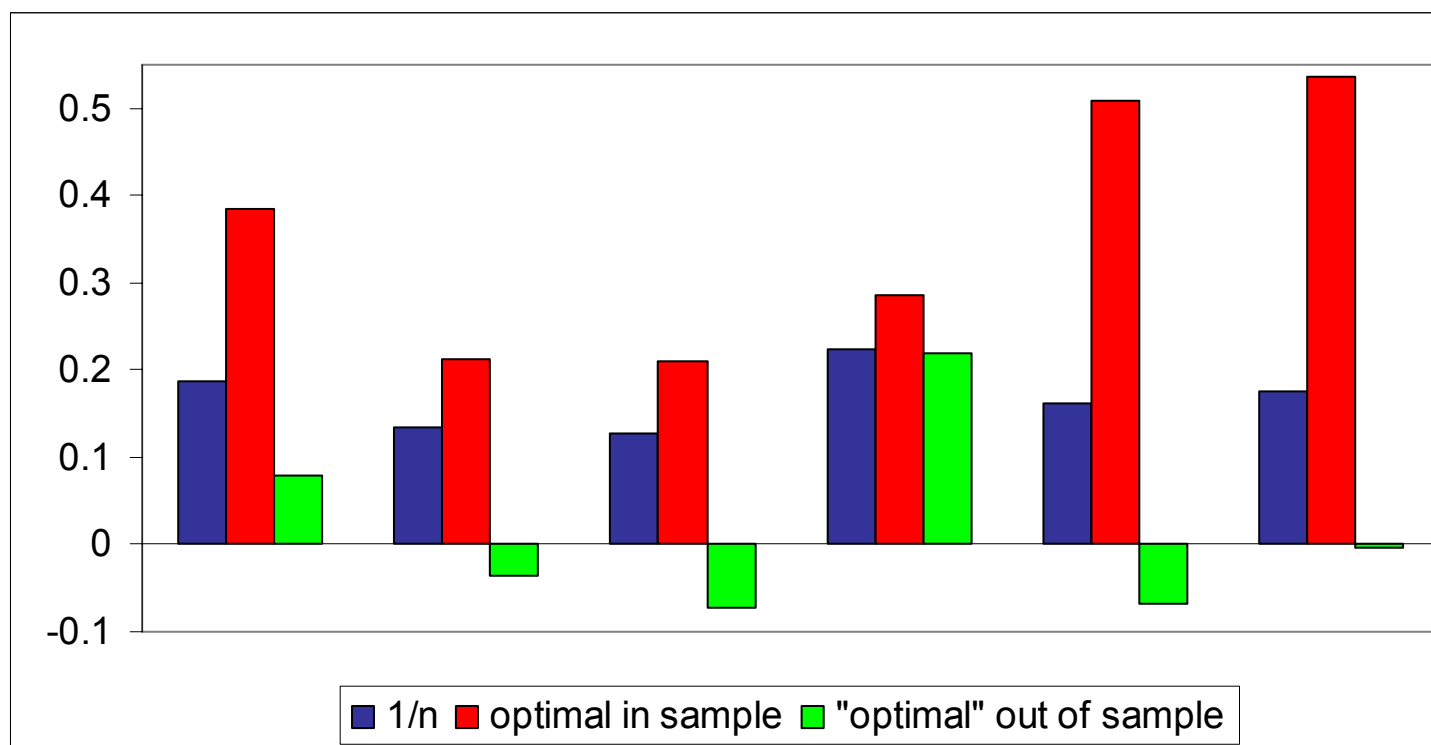


- Three performance measures:
 - out-of-sample Sharpe ratio
 - certainty equivalent return for a mean-variance investor

$$\text{CEQ} = \mu - \frac{\gamma}{2} \sigma^2$$

(γ = absolute risk aversion)

- portfolio turnover (\rightarrow transaction costs)
- Results:
 - Markowitz optimization clearly underperforms the 1/n benchmark



- No model significantly outperforms 1/n
"[O]ur results indicate that, of the various optimizing models in the literature, there is no single model that consistently delivers a Sharpe ratio or a certainty-equivalent return that is higher than that of the 1/N portfolio, which also has a very low turnover" (p. 31/32)

- Interpretation:
 - $1/n$ does not rely on parameter estimates
 - estimation error outweighs the benefits from optimal diversification
 - How long an estimation window would we need in order to reduce estimation error sufficiently?
3000 [months] for a portfolio consisting of 25 [50] assets
(and this still assumes that the true parameters are constant!)
 - More complex optimization models do not reduce these figures substantially

II.3. Individual Investors' Portfolios

What *should* individual investors do?

- Hold a combination of well-diversified mutual funds and bonds
- Include foreign assets

What *do* individual investors do?

- Portfolios are not well diversified
- they are distinctly different
- and exhibit specific biases
 - home bias
 - disposition effect
 - etc. (cf. the behavioral finance literature)

Goetzmann and Kumar (2004) analyze 40,000 retail investor accounts at a discount brokerage firm

- Data allows analysis of both the degree of diversification and the performance of the portfolios
- but does not control for non-financial assets (e.g. real estate) or multiple brokerage accounts

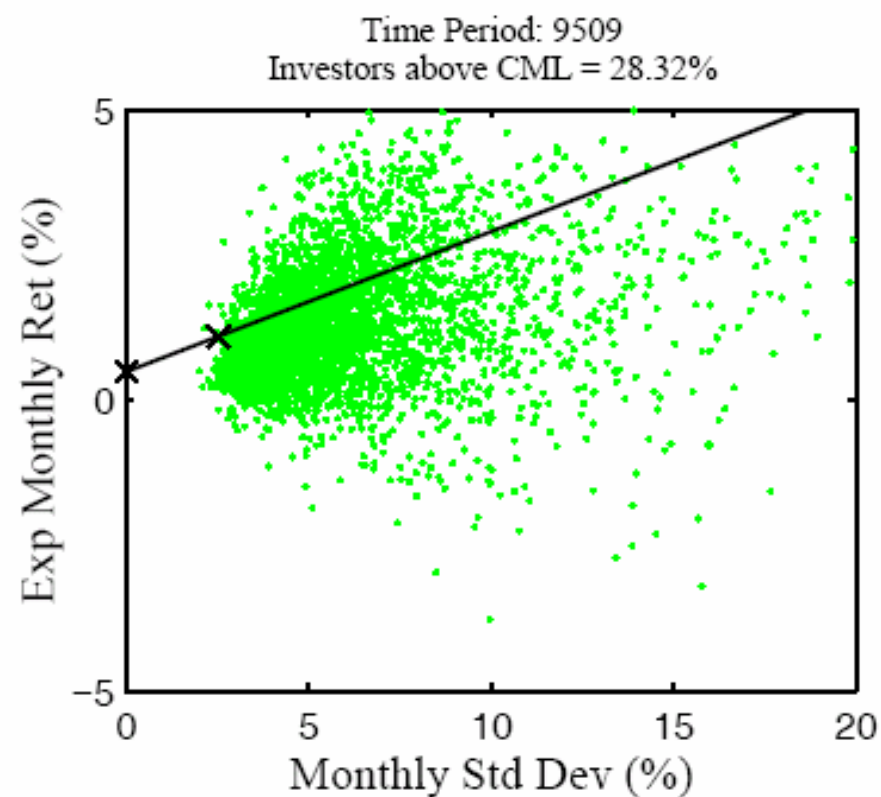
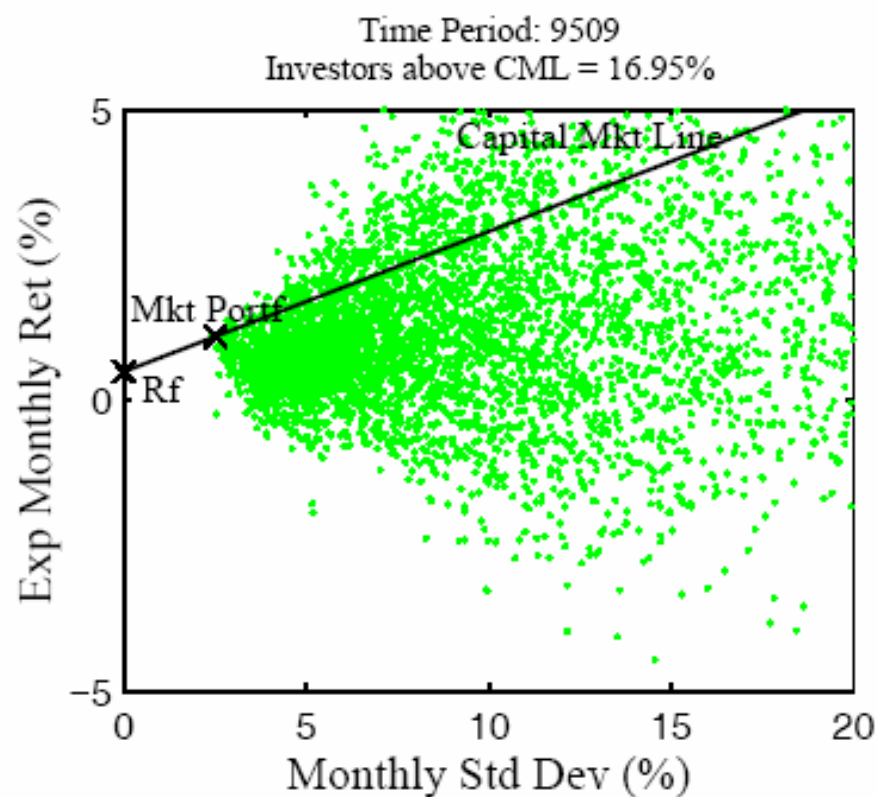
Results:

- Bias towards "household names"
- Low degree of diversification: mean 4 different stocks (median: 3)
- If portfolios are diversified then they are naively diversified
- Older investors have better diversified portfolios
- Investors who trade more frequently have less diversified portfolios

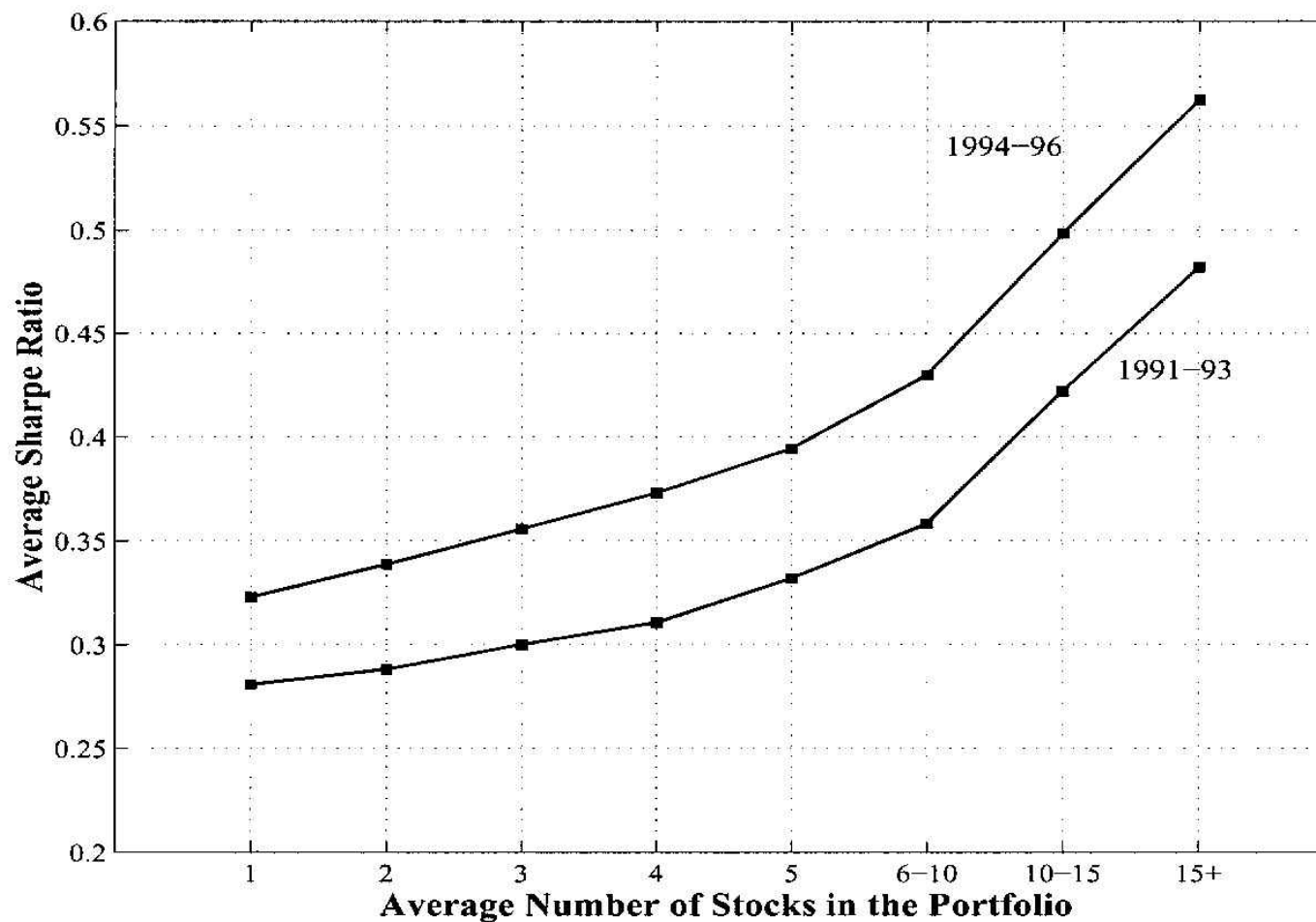
Diversification und Performance (Goetzman / Kumar 2004, Fig. 6+7)

• Ex-ante:

Portfolios with 1-3 stocks (l.) and t or more stocks (r.)



- ex-post [Sharpe-measure]



- Result: Diversified portfolios have better performance both ex-ante and ex-post

Trading is hazardous to your wealth (Barber and Odean 2000)

	Turnover Quintile				
monthly (excess) returns	1 (low)	2	3	4	5 (high)
average return	1,470	1,411	1,361	1,267	1,009
index-adjusted	0,050	-0,009	-0,059	-0,153	-0,411
market model-adjusted	0,077	-0,038	-0,140	-0,314	-0,692
Fama-French-adjusted	-0,061	-0,130	-0,269	-0,464	-0,864

II.4. Do Individual Errors Cancel Out?

Two observations:

- Prices are broadly consistent with equilibrium models like the CAPM
- Individual portfolio holdings grossly deviate from the predictions of theory

Puzzle: "how can pricing theory be right if the portfolio choice theory on which it rests is wrong?" (Bossaerts et al. 2005)

Experimental approach

- Expected returns, variances and covariances are *known*
- Homogeneous expectations can be implemented
- One-period economies can be "created"
- Individual portfolios are observed

The experiments of Bossaerts et al. (2005):

- Static replications of one-period economies
- 2 risky and one riskless asset
- If mean-variance utility is assumed, two-fund separation and the CAPM should hold

Testable hypotheses:

- Each trader holds the market portfolio
- The market portfolio is mean-variance efficient
- No hypotheses on prices (because the market risk premium depends on aggregate risk aversion which is not known)

Results:

- The market portfolio is close to efficient (evaluation based on Sharpe ratios)
- Portfolio holdings are add odds with the predictions
- No relation (!) between the distance from theoretical to actual portfolio holdings and the deviations from the CAPM

One potential explanation:

- Individual demand functions are perturbed
→ deviations from two-fund separation
- Perturbations have zero mean and are independent across individuals
- Then they cancel out in a large market and the CAPM may obtain

Bossaerts et al. (2005) develop a structural model along these lines ("CAPM + ε ") and find empirical support for it in their experimental data

III. Tests of the CAPM

Outline:

1. Preliminaries
2. Estimating Beta
3. The Fama and MacBeth (1973) 3-Step Procedure
4. The Black, Jensen and Scholes (1972) Test
5. Gibbons' Test of the Zero-Beta CAPM
6. The Conditional Relation Between Return and Beta
7. A Test Using Expected Returns
8. Is the CAPM Testable? The Roll (1977) Critique

III.1. Preliminaries

Testable (?) hypotheses:

- Stocks with higher risk (beta) have higher expected return
- The risk-return relation is linear
- Unsystematic risk does *not* affect expected returns
- In a regression of the risk premia of stocks / portfolios on the market risk premium, the intercept α_i should be zero

$$\left(r_{i,t} - r_{f,t} \right) = \alpha_i + \beta_i \left(r_{m,t} - r_{f,t} \right)$$

- Intercept and slope of the security market line are equal to the riskless rate and the market risk premium, respectively (applies to the standard CAPM)
- The market portfolio is on the efficient frontier

Expected versus realized returns:

- CAPM: expected returns

$$\mu_i = r_f + \beta_i (\mu_m - r_f)$$

- Empirical tests are usually based on realized returns

$$(r_{i,t} - r_{f,t}) = \alpha_i + \beta_i (r_{m,t} - r_{f,t}) + \varepsilon_{i,t}$$

(tests are usually conducted on the basis of risk premia)

- Implicit assumptions:
 - rational expectations (expectations are, on average, correct)
or (Elton et al. 2002, p. 339)
 - The market model holds in each period (i.e., the return generating process is a linear one-factor model)
 - The CAPM holds in each period
 - Betas are constant

Classic test procedure (Fama and MacBeth 1973):

- Estimate betas from time-series regression

$$\left(r_{i,t} - r_{f,t} \right) = \alpha_i + \beta_i \left(r_{m,t} - r_{f,t} \right) + \varepsilon_{i,t}$$

- Perform a cross-sectional test of the relation between beta and (realized) returns
- Test whether other variable systematically affect returns

III.2. Estimating Beta

Usual approach: Time-series regression

$$r_{i,t} = \alpha_i + \beta_i r_{m,t} + \varepsilon_{i,t}$$

or

$$(r_{i,t} - r_{f,t}) = \alpha_i + \beta_i (r_{m,t} - r_{f,t}) + \varepsilon_{i,t}$$

- Estimation period: often 60 months
- Assumes constant betas

The OLS estimate $\hat{\beta}_i = \frac{\sigma_{i,m}}{\sigma_m^2}$ is analogous to the CAPM beta

Problems

- Instability of estimated betas over time because of
 - changes in the true beta (structural break)
 - estimation error
- Betas estimated for portfolios are more stable because both effects tend to cancel out
 - ⇒ Many empirical tests of asset pricing models use portfolios rather than individual stocks
- Mean reversion in estimated betas

Fundamental Betas

- Idea: systematic risk of a stock should depend on fundamentals of the firm
- Changes in these fundamentals affect historical betas only very gradually
- Approach: Estimate relation between fundamentals and beta directly:
- First step: cross-sectional regression

$$\beta_i = a + \sum_k b_k X_{i,k} + \eta_i$$

β_i : historical beta

X_i : fundamentals (e.g. capital structure, dividend yield, variability of earnings etc.)

- Second step: obtain beta from observed (or estimated) fundamentals

$$\hat{\beta}_i = a + \sum_k b_k \hat{X}_{i,k}$$

Characterization:

- General superiority over historical betas not established
- In practice, combinations between historical betas and fundamental betas are used (e.g. by BARRA)

A simulation experiment:

Generate 120 (monthly) market returns:

$$r_{m,t} = 0.005 + 0.04 \cdot \text{rand}$$

Generate 120 (monthly) stock returns:

$$r_{i,t} = r_{m,t} + 0.04 \cdot \text{rand}$$

(The true beta is 1)

Estimate the regression

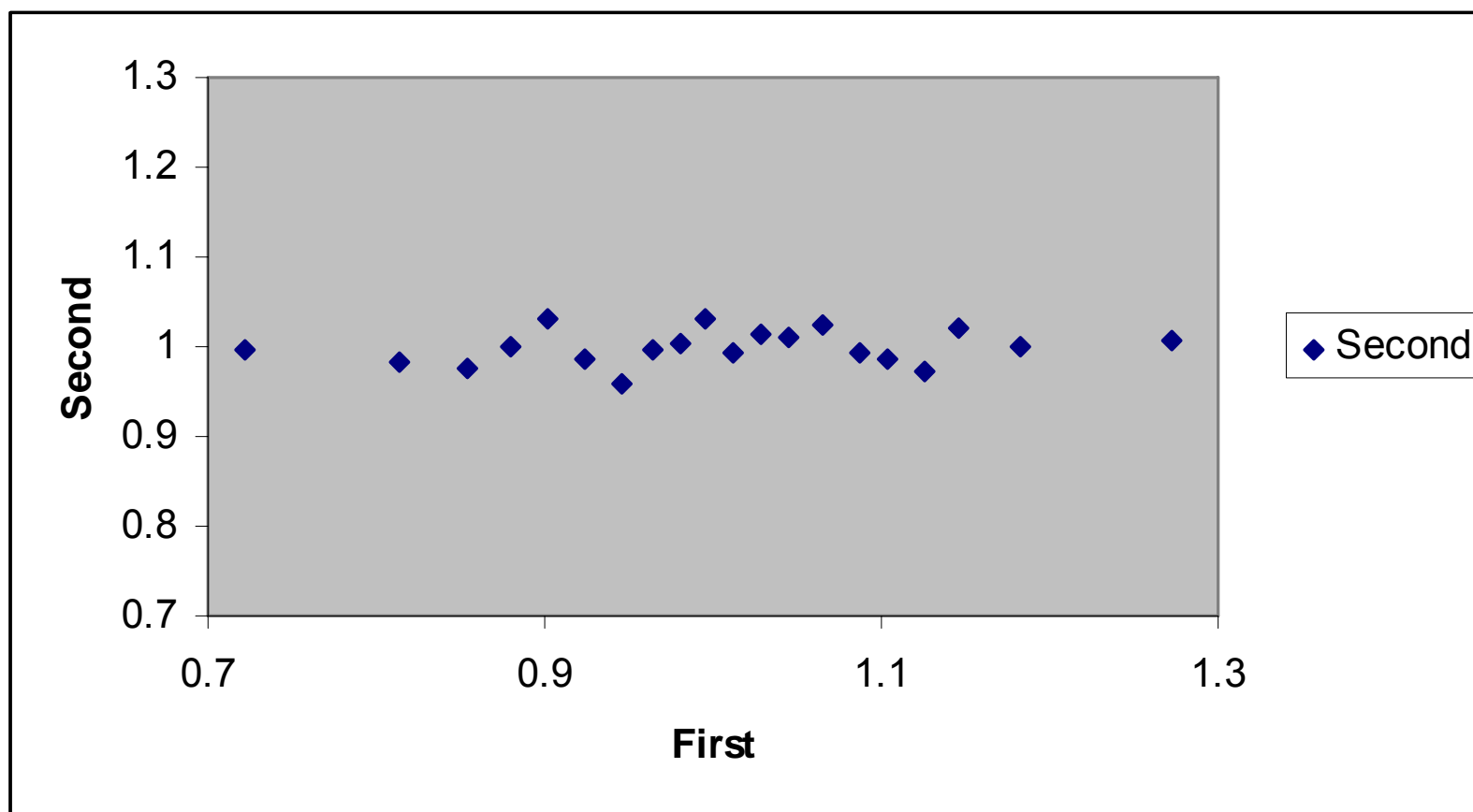
$$r_{i,t} = \alpha_i + \beta_i r_{m,t} + \varepsilon_{i,t}$$

for the first and the last 60 observations

Repeat this procedure 1000 times

Now:

- Sort the 1000 "stocks" into 20 portfolios according to their beta estimate in the first subsample
- Calculate average portfolio betas for the second subsample



Conclusion:

- Stocks are basically sorted with respect to their estimation error
- This justifies the 3-step procedure proposed by Fama and MacBeth (1973)

Estimating beta for infrequently traded stocks

Illiquid stocks do not trade every day. Consequences:

- Daily index returns will exhibit positive serial correlation
- The variance of index returns will be understated
- Covariances between stock and index returns will be biased downwards; the bias depends on the liquidity of the stock
- Estimated beta coefficients depend on the measurement interval (intervalling effect)
- Beta estimates will be biased

Corrections:

- Dimson (1979)
 - Regress stock returns on current, lagged and leading market returns
 - Sum the slope coefficients
- Scholes and Williams (1977):
 - Regress stock returns on current, lagged and leading market returns in separate regressions
 - Obtain $\hat{\beta} = [\beta^{-1} + \beta^0 + \beta^{+1}] / (1 + 2\rho)$ where ρ is the first order serial correlation of the index returns
- Fowler and Rorke (1983)
 - Shows that the Dimson procedure is incorrect (but can be adjusted by using a weighted sum of the coefficients)

- Changes in the betas are trivial when the serial correlation of the index returns is low

Application:

- Fama and French (1992) use Dimson betas
- Event studies (often use beta estimates from daily data)

III.3. The Fama and MacBeth (1973) 3-Step Procedure

Step 1: Portfolio formation

- Estimate beta for individual stocks from time-series regression (e.g. 4 years of monthly data)
- Sort stocks into portfolios according to the estimated beta (e.g. 20 portfolios)
- Why portfolios?
 - measurement error
 - changes in true betas

Step 2: Estimate portfolio betas

- Estimate betas for the portfolios using data from a distinct sample period
- Avoids the measurement error bias discussed earlier

Step 3: Test the CAPM

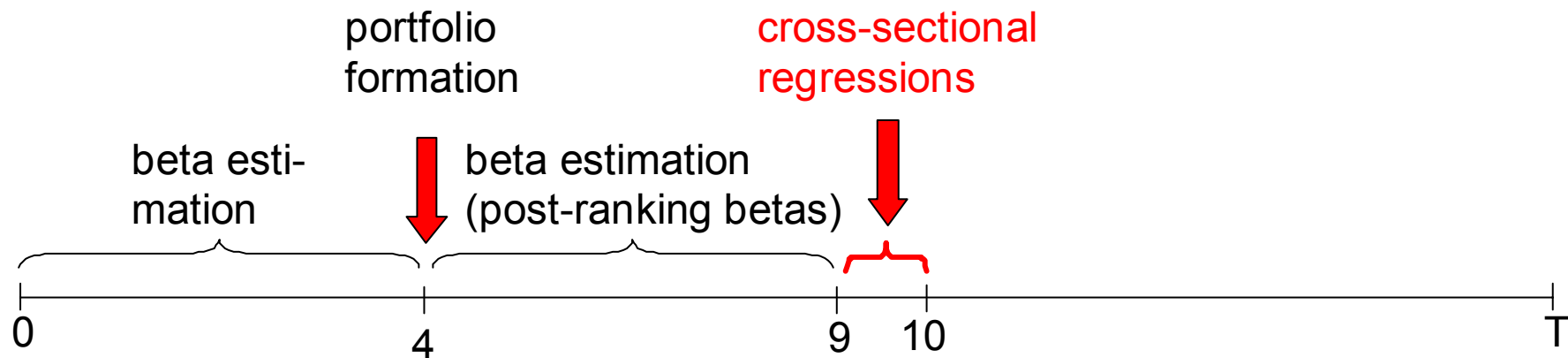
- Test the risk-return relation in monthly cross-sectional regressions

$$r_i = \gamma_0 + \gamma_1 \beta_i + \eta_i$$

- t-tests of $\gamma_0 ; \gamma_1$ will be misspecified when there is cross-sectional dependence (i.e. when $\text{Cov}(\eta_i, \eta_j) \neq 0 ; i \neq j$), therefore:
- Calculate and test the time-series averages of γ_0, γ_1 (simple t-tests)

- The CAPM predicts $\gamma_0 = \bar{r}_f$; $\gamma_1 = (\bar{r}_m - \bar{r}_f)$
Often only $\bar{\gamma}_1 > 0$ is tested
- Test whether other variables affect returns, e.g.:

$$r_i = \gamma_0 + \gamma_1 \beta_i + \gamma_2 \beta_i^2 + \gamma_3 S_{\varepsilon i} + \eta_i$$



Results (Fama and MacBeth 1973, p. 622/623; USA, 1926-1968):

γ_0	γ_1	γ_2	γ_3
0,0061 (3,24)	0,0085 (2,57)		
0,0049 (1,92)	0,105 (1,79)	-0,0008 (-0,29)	
0,0054 (2,10)	0,0072 (2,20)		0,0198 (0,46)
0,020 (0,55)	0,0114 (1,85)	-0,0026 (-0,86)	0,0516 (1,11)

- Beta is the only systematic factor
- The intercept tends to be "too large" and the slope "too low". This is inconsistent with the standard CAPM but consistent with the zero-beta CAPM

Problems:

- Errors-in-variables problem
 - Remedy 1: use portfolios rather than individual stocks
 - Remedy 2: explicitly adjust for the EIV problem (Litzenberger and Ramaswamy 1979, Shanken 1992). It entails subtracting a correction factor from the cross-product matrix of the estimated betas
 - In practical applications, the Shanken correction often yields a modified cross-product matrix that is not positive definite as it should be (Shanken and Weinstein 2006)
 - Chen and Kan (2004) propose an alternative adjustment procedure and use simulations to show their finite-sample properties

- The true market portfolio is mean-variance efficient under the CAPM, but the empirical proxy is not. Small deviations from efficiency can have a great impact on the cross-sectional relation between beta and return (Roll and Ross 1994).

III.4. The Black, Jensen and Scholes (1972) Test

- The test makes use of the fact that under the CAPM the intercept α_i should be zero in

$$(r_{i,t} - r_{f,t}) = \alpha_i + \beta_i (r_{m,t} - r_{f,t})$$

- Black et al. group the stocks into 10 portfolios (using pre-test period data as in Fama and MacBeth 1973)
- They then estimate $(r_{i,t} - r_{f,t}) = \alpha_i + \beta_i (r_{m,t} - r_{f,t})$ and test α_i against zero
- Result: High [low] beta stocks tend to have negative [positive] alphas; 3 out of 10 coefficients are significant
- These results are inconsistent with the standard CAPM but consistent with the zero-beta CAPM

Extending the BJS Approach

Campbell et al. (1997) propose to estimate

$$\mathbf{r}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} \mathbf{r}_{m,t} + \boldsymbol{\varepsilon}_t$$

using ML (alternatively: GMM which is robust under non-normality) and then to test the restriction $\boldsymbol{\alpha} = \mathbf{0}$

$[\mathbf{r}_t ; \mathbf{r}_{m,t} = r_{m,t} \mathbf{1}]$: vector of excess returns / of market returns]

They propose four different test statistics:

- a Wald test (J_0)
- a finite-sample F-test (J_1) [due to Gibbons et al. 1989]
- a likelihood ratio test (J_2)
- an adjustment of the LR test with better finite-sample properties (J_3)

Notwithstanding these approaches, the cross-sectional (Fama and MacBeth-type) tests are more frequently used

A Reinterpretation:

- When the market portfolio is ex-post efficient, a linear relation between stock returns and the market return must hold (see section III.8)
- Therefore, the tests are paramount to testing whether the market portfolio is efficient.
- This, in turn, is equivalent to testing whether the market portfolio has maximum Sharpe Ratio among all portfolios

III.5. Gibbons' Test of the Zero-Beta CAPM

The test uses the fact that the model implies

$$\boldsymbol{\alpha} = (\mathbf{1} - \boldsymbol{\beta}) \lambda$$

It tests the unrestricted model:

$$\mathbf{r}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} \mathbf{r}_{m,t} + \boldsymbol{\varepsilon}_t$$

against the restricted model:

$$\mathbf{r}_t = (\mathbf{1} - \boldsymbol{\beta}) \lambda + \boldsymbol{\beta} \mathbf{r}_{m,t} + \boldsymbol{\varepsilon}_t$$

using a LR test

Details:

- ML estimates must be obtained iteratively because the restriction $\boldsymbol{\alpha} = (\mathbf{1} - \boldsymbol{\beta}) \lambda$ is non-linear

- Alternatively, one can use a one-step estimator and linearize the restriction using a Taylor approximation (the original approach in Gibbons 1982)
- Note that seemingly unrelated regression is not an improvement (as the regressors are identical in all equations)
- It is unnecessary to include additional variables in the regression (as in Fama and MacBeth 1973) - if other variables would systematically affect returns, the restriction $\alpha = (\mathbf{1} - \beta)\lambda$ would be rejected

Result:

- Gibbons estimated his model on 40 beta-sorted portfolios
- He rejects the CAPM in 5 out of 10 subperiods

Problems:

- The iterative estimation procedure
- Test is based on large-sample theory
Modified tests with better small sample properties have been developed by Kandel (1984) and Shanken (1986)
- Shanken and Zhou (2006) provide a simulation-based analysis of different testing approaches. No clear winner is identified:
"Since no single estimation procedure dominates in all respects, it might be wise to explore robustness of results to several estimation approaches in applied work" (p. 27)

III.6. The Conditional Relation Between Return and Beta

The joint hypothesis problem:

- CAPM: *Expected* returns:

$$E(r_i) = r_f + \beta_i [E(r_m) - r_f]$$

- Empirical tests: *Realized* returns:

$$r_i = r_f + \beta_i [r_m - r_f] + \varepsilon_i$$

- The *expected* market risk premium is positive. With respect to the realized market risk premium there is a problem: From $r_i = r_f + \beta_i [r_m - r_f] + \varepsilon_i$ it follows that $\beta_i > \beta_j \Leftrightarrow r_i > r_j$ if the market risk premium is positive and $\beta_i > \beta_j \Leftrightarrow r_i < r_j$ if the market risk premium is negative.

A thought experiment:

- Assume the CAPM holds in each month but the risk premium is negative in half of the months
- Then there will be no relation between beta and return overall although there is a relation in each individual month

Put differently: If no relation between beta and return is found, there may be two reasons for this result

- There is no such relation
- The market risk premium was non-positive in the sample period

Thus: Traditional CAPM tests test the *joint hypothesis* that the CAPM holds and that the market risk premium is positive

Example DAFOX 1960 - 1995: 48,1% of the monthly risk premia were negative

A Modified test (Pettengill et al. 1995)

Since $\beta_i > \beta_j \Leftrightarrow r_i > r_j$ if the market risk premium is positive and $\beta_i > \beta_j \Leftrightarrow r_i < r_j$ if the market risk premium is negative, the relation between beta and return is conditional on the sign of the market risk premium. A simple way to incorporate this into the second-pass regression is

$$r_i = \gamma_0 + \gamma_1 D \beta_i + \gamma_2 (1 - D) \beta_i + \varepsilon_i$$

with the hypothesis

$$\gamma_1 > 0 ; \gamma_2 < 0$$

Some Results for Germany (Elsas et al. 2003)

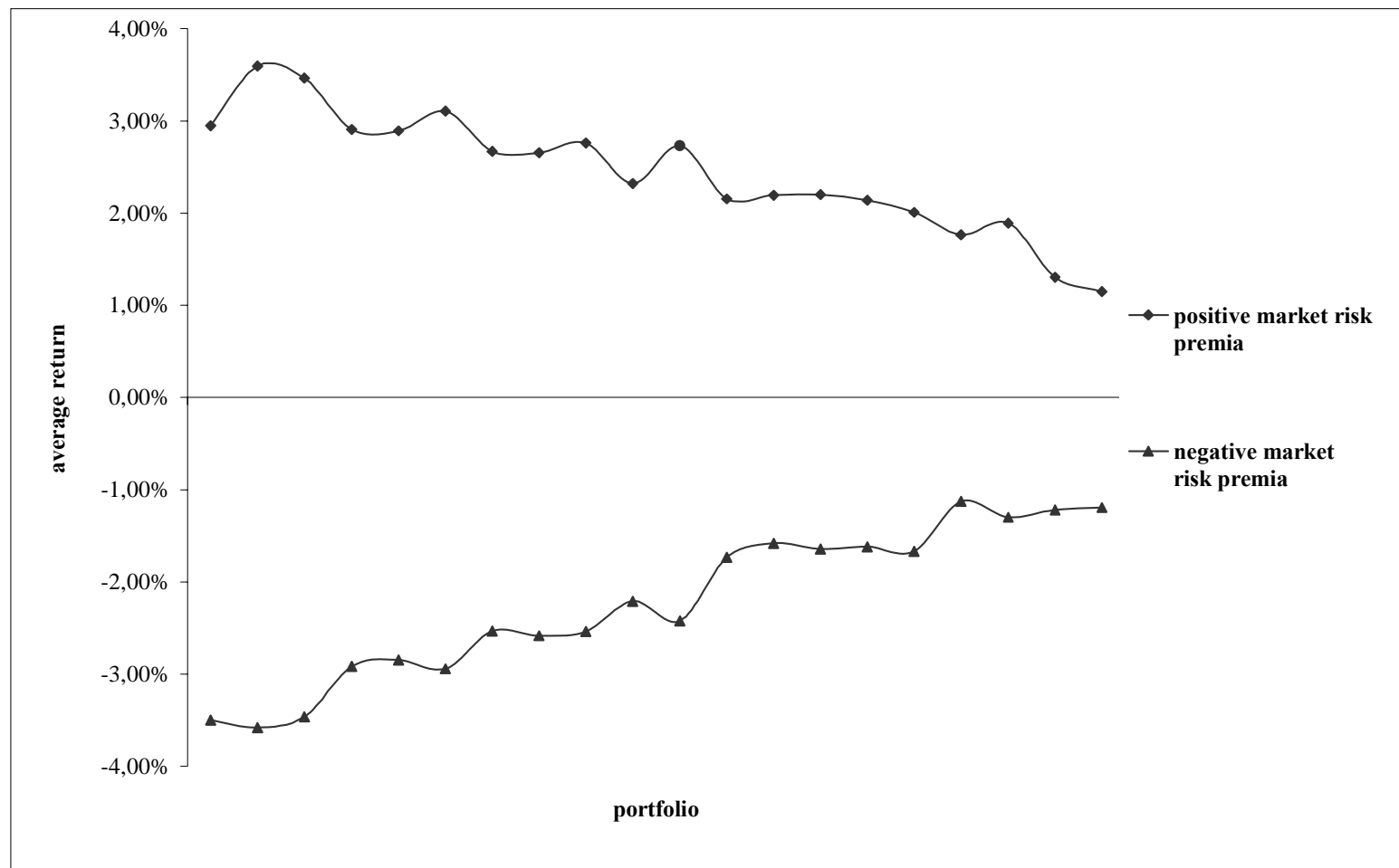
Data: Monthly returns 1960-1995

A: Traditional test: $r_i = \gamma_0 + \gamma_1 \beta_i + \eta_i$ with the hypothesis $\gamma_1 = 0$

period	$\hat{\gamma}_1$	p-value
1968-1995	0,22%	0,366
1968-1976	-0,19%	0,637
1977-1985	1,00%	0,005
1986-1995	-0,11%	0,824

B: Modified test $r_i = \gamma_0 + \gamma_1 D \beta_i + \gamma_2 (1 - D) \beta_i + \varepsilon_i$ with the hypotheses $\gamma_1 > 0 ; \gamma_2 < 0$

period	positive market risk premium		negative market risk premium	
	$\bar{\hat{\gamma}}_1$	p-value	$\bar{\hat{\gamma}}_2$	p-value
1968-95	2,79%	0,000	-2,71%	0,000
1968-76	2,47%	0,000	-2,58%	0,000
1977-85	2,95%	0,000	-1,74%	0,000
1986-95	2,89%	0,000	-3,65%	0,000



Result and conclusions:

- There is a significant relation between systematic risk and return
- Using beta in portfolio management is justified

How to Interpret these Results?

Consider the return generating process (Cooper 2006)

$$r_{i,t} = E(r_m) + \beta_i [r_{m,t} - E(r_m)] + \varepsilon_{i,t}$$

- In a cross-sectional regression (realized) returns will clearly be related to betas
- If beta was measured without error we would have
$$\gamma_1 = E[r_m | r_m > E(r_m)] - E(r_m) > 0; \gamma_2 = E[r_m | r_m < E(r_m)] - E(r_m) < 0$$
- But the CAPM is wrong since the return generating process implies that expected returns are unrelated to beta
- Conclusion: The Pettengill et al. procedure tests whether beta is related to (realized) returns but it does not test whether beta risk is priced

Pettengill et al. propose to test

1. whether the mean market risk premium is positive and
2. whether $\gamma_1 = -\gamma_2$ or equivalently whether $\gamma_1 + \gamma_2 = 0$

Rejection of (1) would be a problem for any asset pricing model

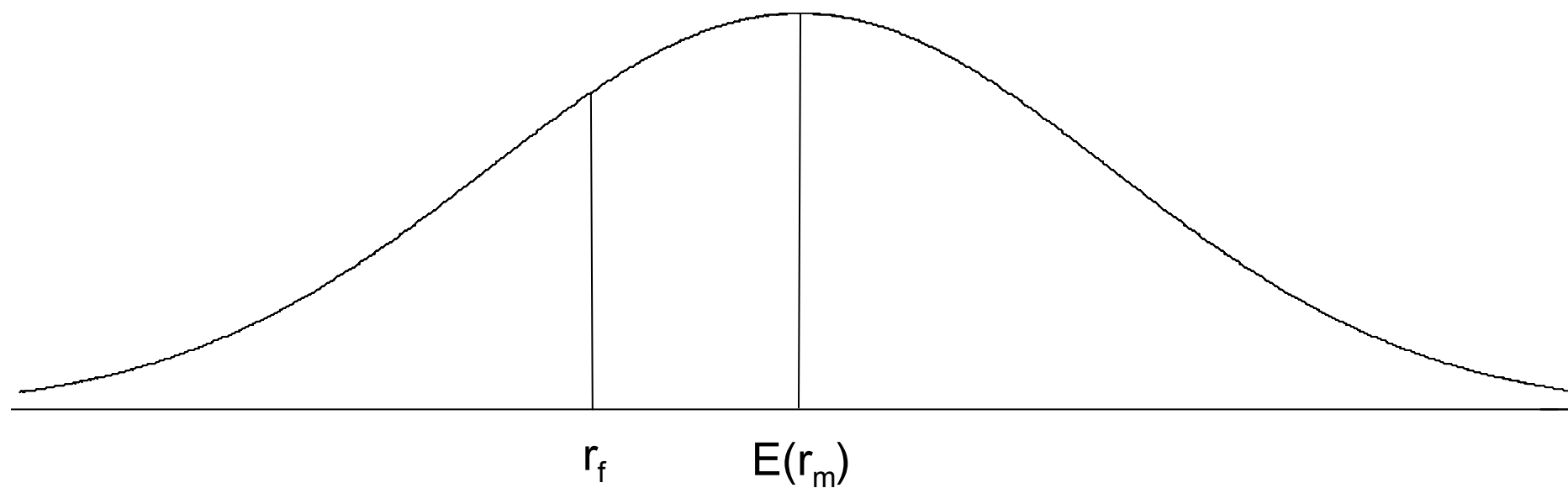
(2) is meant to test whether there is "a consistent relationship between risk and return during up markets and down markets" (Pettengill et al. (1995), 113)

Under the CAPM we would have

$$\gamma_1 = E[r_m | r_m > r_f] > 0; \gamma_2 = E[r_m | r_m < r_f] < 0$$

However, even with a symmetric distribution of the market risk premium, we have

$$E[r_m | r_m > r_f] \neq -E[r_m | r_m < r_f] \text{ if } E(r_m > r_f)$$



Thus, the hypothesis $\gamma_1 + \gamma_2 = 0$ is misspecified (Freeman and Guermat 2006)

III.7. A Test Using Expected Returns (Brav et al. 2005)

- Idea: Relate betas estimated in a first-pass regression to *expected* returns
- Expected returns are derived from analysts' target price estimates
- Procedure similar to Fama and MacBeth (1973) and Fama and French (1992)
- Size and book-to-market are included in the second-pass regression

	Size / BM measured by mimicking portfolios		Size / BM measured by firm characteristics
	Modell 1	Modell 2	Modell 3
Konstante	0.068	0.068	0.119
Beta	0.067	0.058	0.078
Size		0.016	-0.010
Book-to-market		0.002	0.012
R ²	0.06	0.12	0.13

Brav et al. (2005), Table III; a shaded cell indicates significance at the 5%-level

III.8. Is the CAPM Testable? The Roll (1977) Critique

Roll (1977):

- If the proxy for the market portfolio is ex-post efficient, then there is a linear relation ex post between the returns on the stocks that constitute the proxy and the betas measured relative to the proxy
- Thus, finding a linear relation between beta and return only indicates that the market model proxy was ex-post efficient.

Proof (following Levy 1983, p. 146): Consider the portfolio optimization problem using **realized** returns (i.e., the identification of an ex-post efficient portfolio)

$$\text{Var}(\tilde{r}_p) = \sum_{i=1}^m \sum_{k=1}^m x_i x_k \sigma_{i,k} \rightarrow \text{Min}$$

$$\text{s.t. } \sum_{i=1}^m x_i r_i + \left(1 - \sum_{i=1}^m x_i\right) r_f = \bar{r}$$

we have

$$\bar{r} = \sum_{i=1}^m x_i r_i + r_f - \sum_{i=1}^m x_i r_f \quad \text{und} \quad (\bar{r} - r_f) = \left(\sum_{i=1}^m x_i r_i - \sum_{i=1}^m x_i r_f \right)$$

The Lagrangian is

$$L = \sum_{i=1}^m \sum_{k=1}^m x_i x_k \sigma_{i,k} + \lambda \left[\bar{r} - \sum_{i=1}^m x_i r_i - \left(1 - \sum_{i=1}^m x_i\right) r_f \right]$$

First order condition:

$$2 \sum_{k=1}^m x_k \sigma_{i,k} - \lambda r_i + \lambda r_f = 0 \Rightarrow \frac{\lambda}{2} = \frac{\sum_{k=1}^m x_k \sigma_{i,k}}{(r_i - r_f)}$$

Multiplying by x_i :

$$x_i \frac{\lambda}{2} = \frac{x_i \sum_{k=1}^m x_k \sigma_{i,k}}{(r_i - r_f)} \Rightarrow x_i (r_i - r_f) \frac{\lambda}{2} = x_i \sum_{k=1}^m x_k \sigma_{i,k}$$

Summing over the m first order conditions yields:

$$\sum_{i=1}^m x_i (r_i - r_f) \frac{\lambda}{2} = \sum_{i=1}^m x_i \sum_{k=1}^m x_k \sigma_{i,k} \Rightarrow \frac{\lambda}{2} \left(\sum_{i=1}^m x_i r_i - \sum_{i=1}^m x_i r_f \right) = \sum_{i=1}^m \sum_{k=1}^m x_i x_k \sigma_{i,k}$$

$$\frac{\lambda}{2} (\bar{r} - r_f) = \sigma_p^2 \Rightarrow \frac{\lambda}{2} = \frac{\sigma_p^2}{(\bar{r} - r_f)}$$

Substituting back into the foc for asset i :

$$\frac{\sigma_p^2}{(\bar{r} - r_f)} = \frac{\sum_{k=1}^m x_k \sigma_{i,k}}{(r_i - r_f)}$$

Recognizing that $\sum_{k=1}^m x_k \sigma_{i,k} = \sigma_{i,p}$, we have

$$\frac{\sigma_p^2}{(\bar{r} - r_f)} = \frac{\sigma_{i,p}}{(r_i - r_f)} \Rightarrow r_i = r_f + \frac{\sigma_{i,p}}{\sigma_p^2} (\bar{r} - r_f) = r_f + \beta_p (\bar{r} - r_f)$$

The portfolio characterized by $[\bar{r}; \sigma_p^2]$ is any(!) ex-post efficient portfolio. Irrespective of which ex-post efficient portfolio is chosen we obtain a linear relation between the return of the constituent stocks and their betas measured relative to the ex-post efficient portfolio

Conclusion:

- If the market proxy used in an empirical analysis is ex-post efficient, we will necessarily find a linear relation between beta and return. Thus:
 - Supportive test results do not imply that the CAPM holds
- If the CAPM holds but the market proxy is inefficient, the test may reject the CAPM. Thus:
 - Negative test results do not imply that the CAPM is wrong

What testable hypothesis, then, does the CAPM imply?

- The market portfolio is ex-post efficient
- Problem: The market portfolio is unobservable

Living with the Roll Critique:

1. Use more comprehensive proxies for the market portfolio (Stambaugh 1982). This does not solve the problem in a strict sense. However: "and thus Roll's concern is not an empirical problem" (Campbell et al. 1997, 215)
2. Even if the proxy is not identical to the true market portfolio, the two should be highly correlated. One can test simultaneously whether
 - the CAPM holds with respect to the unobservable true market portfolio and
 - whether the correlation between the market proxy and the true market portfolio exceeds a given threshold.

Shanken (1987) uses this approach and rejects the CAPM. The correlation between the market proxy and the true market portfolio would have to be below 0.7 for the CAPM not to be rejected.

IV. Testing the APT

Tests of the APT are complicated by the fact that the theory does not specify the factors. There are three approaches:

1. Simultaneous estimation ("extraction") of factors and sensitivities ("factor loadings") using multivariate methods (factor analysis)
2. Firm characteristics (e.g. size, book-to-market) are interpreted as factor sensitivities, risk premia are estimated in cross-sectional regressions
3. Factors are specified and sensitivities are estimated
 - a) factor-mimicking portfolios
 - b) macro variables

Factor Analysis

Characterization:

- Both factors and factor loadings are extracted from the data
- Useful to determine the number k of factors
LR test of the null that k factors are appropriate
Roll and Ross (1980) conclude that 3-4 factors are appropriate
- Problem 1: Number of factors tends to increase in the number of stocks in the sample
- Problem 2: Factors are not identified as economic variables
- Problem 3: Factors are non-unique linear combinations of the true (unobservable) factors

Lehman and Modest (1988)

- Estimate factor sensitivities using ML factor analysis
Daily data 1963-82, 750 securities
5, 10, 15 factors (i.e., # of factor is exogenous)

- Construct the basis portfolios from the estimates

$$\min_{\mathbf{x}_j} \mathbf{x}'_j \mathbf{D} \mathbf{x}_j \quad \text{s.t.} \quad \mathbf{x}'_j \mathbf{b}_k = 0 \quad \forall j \neq k \quad \wedge \quad \mathbf{x}'_j \mathbf{1} = 1$$

D: diagonal matrix of residual variances,

x_j: weights of basis portfolio for factor *j*

b_k: factor loadings for factor *k*

- Test whether the APT can account for "anomalies" in cross-sectional regressions (weekly data)
 - sort stocks into 20 equally-weighted (e.g.) size portfolios

- regress weekly excess returns on the factor risk premia (i.e., the returns on the basis portfolios)
- test the restriction that the intercept is zero in all 20 equations
- Interpretation: If the APT holds than the factor sensitivities and risk premia should explain the excess returns of the size portfolios (i.e., there should be no systematic differences in excess returns that are left unexplained)
- Results:
 - APT does not explain the size anomaly
 - It does explain the relation between returns and dividend yield or own variance

An alternative to factor analysis is principal components analysis (Conner and Korajczyk 1986, 1988)

Pre-specified factor sensitivities

- Firm characteristics are interpreted as factor sensitivities
- Relegated to chapter VI (e.g. Fama and French 1992)

Pre-specified factors - mimicking portfolios

- Portfolio returns are interpreted as factor realizations
- Relegated to chapter VI (e.g. Fama and French 1993)

Pre-specified factors - macro variables

Procedure:

- Identify a set of macro variables and financial variables that (hopefully) determine asset returns (\rightarrow ad-hoc model)
- Estimate the unexpected component in the factor returns
- Regress asset returns on these unexpected components

Economic forces and the stock market (Chen et al. 1986):

The "model":

$$P_0 = \sum_{t=1}^T \frac{CF_t}{(1+r_t)^t}$$

Returns are determined by factors that

- affect future cash flows
- affect the discount rate

Macro and financial variables:

- growth rate of industrial production (leaded by 1 period)
- unanticipated inflation (were expected inflation is estimated using nominal interest rates)
- changes in the expected rate of inflation

- the risk premium as measured by default spreads
- the term premium as measured by term spreads
- the market return (equally and value-weighted NYSE index)
- changes in real consumption
- changes in oil prices

Method: 2-pass regression à la Fama and MacBeth (1973)

- time series regressions to estimate the sensitivities
- cross-sectional monthly regressions of returns (on 20 size portfolios) on the sensitivities
- test the time series averages of the estimated risk premia against zero

Results:

- Industrial production, changes in the risk and in the term premium are significant
- Both inflation measures have weaker impact
- Real consumption and oil prices are insignificant
- The market return have no explanatory power when the other variables are included
- Shanken and Weinstein (2006) show that the results are very sensitive to small changes in the methods employed.

V. Anomalies or Priced Risk Factors?

Outline:

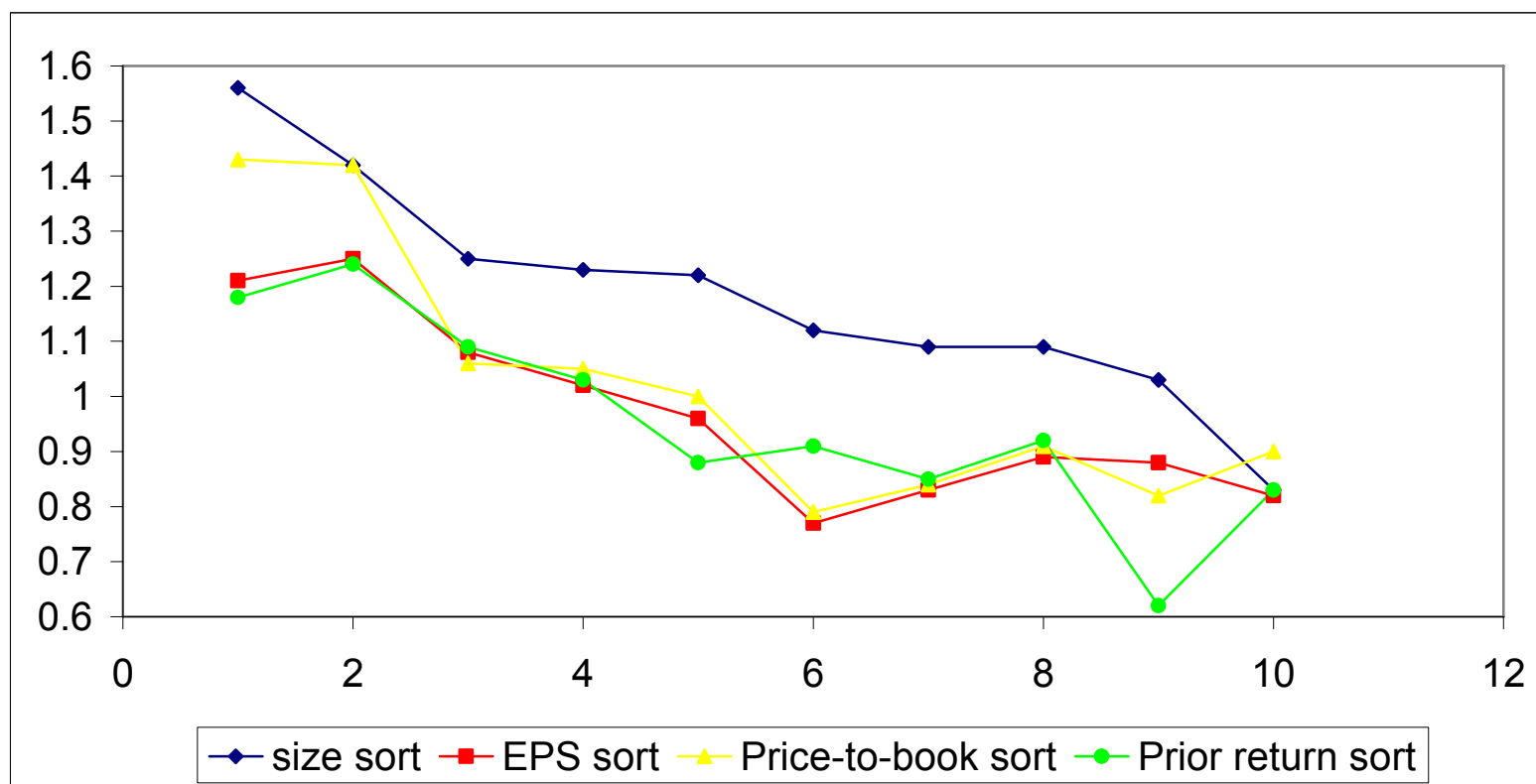
1. The Evidence
2. Interpretation
3. A Skeptical Appraisal

V.1. The Evidence

Starting in the 1980s, several "anomalies" have been discovered, e.g.:

- the size effect
- the earnings per share effect
- the book-to-market effect
- momentum effects

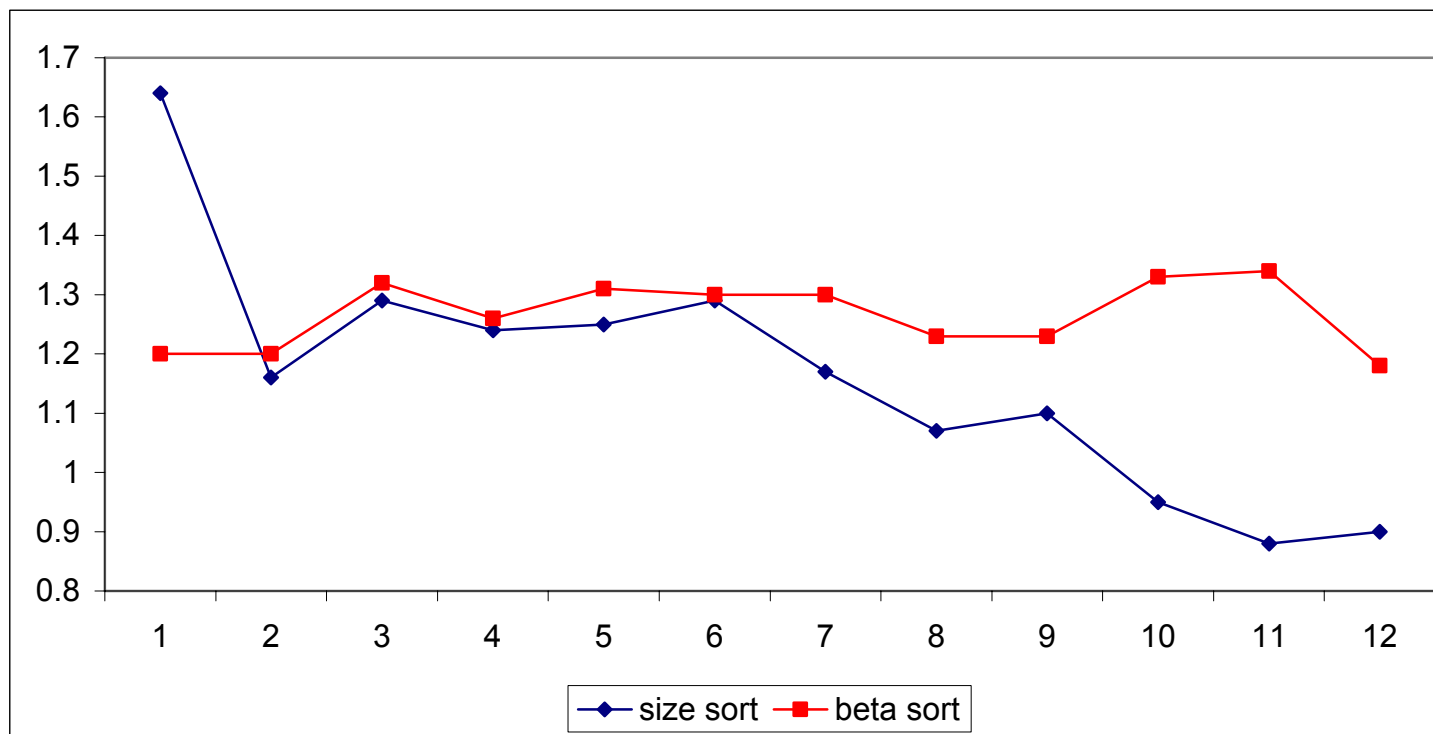
These are inconsistent with the CAPM but potentially consistent with the APT



Figures taken from Hawawini and Keim (1998); USA 1962-1994
(Note that beta is essentially flat across portfolios)

Fama and French (1992) [FF92]

- Empirical analysis based on 10 portfolios sorted by size (sort results in wide spread of average returns and betas)
- Size portfolios broken down in 10 beta portfolios ("size-beta sort") using pre-ranking betas for sorting
- Data for 1963-1990 for the 100 portfolios
- Full-sample Dimson betas (current and lagged month) are used in the cross-sectional analysis (post-ranking 5-year betas yield similar results)
Trade-off: stability of portfolio betas versus precision of estimates
- Results (taken from Table 2 in FF92):



- Strong negative relation between size and return (portfolio 1: smallest firms)
Note though: size and beta are highly correlated, smaller firms have higher betas
- No apparent relation between beta and return (portfolio 1: smallest beta)

- Fama MacBeth-type cross-sectional regressions: Returns are regressed on beta, the log of the market value of equity, the log of the ratio of book value to market value of equity, two leverage variables (book and market leverage), and two earnings per share variables (EPS^+ and a dummy for negative earnings)
- These firm characteristics are interpreted as sensitivities to a (potentially) priced risk factor
- Results (Table 3 in FF92):
 - Beta has no explanatory power whereas size and book-to-market do have considerable explanatory power
 - Leverage does have power (but the difference in the two leverage variables equals $\ln(BE/ME)$); EPS loses power when size and BE/ME are included
- Punchline: "... market β seems to have no role in explaining the average returns ... while size and book-to-market equity capture the cross-sectional variation in average stock returns" (p. 445)

β	ln(ME)	ln(BE/ME)	ln(A/ME)	ln(A/BE)	E/P Dummy	E(+)/P
0.15 (0.46)						
	-0.15 (-2.58)					
-0.37 (-1.21)	-0.17 (-3.41)					
		0.50 (5.71)				
			0.50 (5.69)	-0.57 (-5.34)		
					0.57 (2.28)	4.72 (4.57)
	-0.11 (-1.99)	0.35 (4.44)				
	-0.11 (-2.06)		0.35 (4.32)	-0.50 (-4.56)		
	-0.16 (-3.06)				0.06 (0.38)	2.99 (3.04)
	-0.13 (-2.47)	0.33 (4.46)			-0.14 (-0.90)	0.87 (1.23)
	-0.13 (-2.47)		0.32 (4.28)	-0.46 (-4.45)	-0.08 (-0.56)	1.15 (1.57)

Fama and French (1993) [FF93]

- FF93 explain the cross-section of stock and bond returns:
 - 25 stock portfolios, sorted on size and BE/ME
 - 2 government bond portfolios (sorted by time to maturity)
 - 5 corporate bond portfolio (sorted by rating)
- Explanatory variables are factor-mimicking portfolios (i.e. pre-specified factors):
 - term premium (return long term government bonds - T bills)
 - default spread (return corporate - government bonds)
 - market risk premium (return value-weighted stock portfolio - T bills)
 - size and BE/ME

BE/ME Size	high (top 30%)	medium (40%)	low (30%)
small	SH	SM	SL
large	LH	LM	LL

- Method:
 - Time-series regressions (similar to Black et al. 1972)
 - Slope coefficients interpreted as sensitivities (or factor loadings)
 - Intercept can be tested against zero

- Results:
 - Size and BE/ME are significant
 - Term and default spread do a good job at explaining bond returns
 - Stock and bond markets are linked (i.e., stock market factors help to explain bond returns and vice versa)
 - The intercepts from 3-factor regressions are close to 0 (Gibbons et al. 1989 test procedure). Even though the null is marginally rejected, FF93 argue that their 3-factor model does "surprisingly well" (p. 41).
 - The regression residuals do not exhibit a pronounced January seasonal (i.e., the 3 factors capture the January seasonal)

- In addition, Fama and French (1996) show that their 3-factor model explains the returns on portfolios formed by earnings-price ratios, cash flow-price ratios, sales growth, and the long-term reversal of stock returns (deBondt and Thaler 1985)

V.2. Interpretation

Interpreting the FF Results

- The CAPM is incorrect, size and BE/ME are priced risk factors
- The size and the BE/ME effects are statistical artefacts and do not "really" exist
- The size and the BE/ME are anomalies, i.e., they constitute deviations from market equilibrium

Priced Risk Factors

What is needed is a theory that explains *why* size and BE/ME should be priced risk factors

- Cochrane (1999):
 - Risk implicit in human capital is non-tradable, non-diversifiable and correlated with the stock market
 - If human capital is exposed to recession risk, you want to avoid holding stocks that are very sensitive to recession risk
 - Small firms and high BE/ME firms are often distressed and therefore exposed to recession risk
 - In equilibrium, then, there must be a premium for holding these stocks
- Size and BE/ME may proxy for default risk (Vassalou and Xing 2004)

- Size may proxy for liquidity (Amihud and Mendelson 1986, 1991)

Default Risk (Vassalou and Xing 2004):

- Use Merton (1974) to estimate default probabilities from stock return data (similar to the KMV approach)
- Default risk is related to average returns (sort on default risk)
- Fama MacBeth cross-sectional regressions, performed on individual stocks, not portfolios (Vassalou and Xing 2004, Table 9)

	Constant	DLI	DLI2	Size	Size2	BM	BM2	SizeDLI	BMDLI
Coef	1.3087	-4.8980	17.8748	-0.0030	0.0000	0.5710	-0.0293	-0.6800	0.1071
<i>t</i> -value	4.4352	-2.7120	4.3832	-0.5061	-0.2406	5.5091	-1.5762	-3.8740	1.9802
Coef	1.3027	-6.2470	19.7108	-0.0072	0.0000			-0.7869	
<i>t</i> -value	4.3906	-3.3818	4.6873	-1.1187	0.2159			-4.2910	
Coef	1.2905	0.7063	2.1471			0.5899	-0.0477		0.1345
<i>t</i> -value	4.3421	0.5158	3.6537			5.7721	-2.4581		2.1236

[DLI is the default risk variable, "2" denotes a squared variable, SizeDLI and BMDLI are interaction terms; Size and BM are rendered orthogonal to DLI before estimation]

- Size per se does not play a role, but its interaction with DLI does
- BM and DLI do have explanatory power
- There is evidence on non-linearity
- Punchline: Size and BM are related to default risk, but they contain other relevant information as well

Statistical Artefacts

The fact that size and BE/ME show up significantly is due to measurement error

Two arguments:

- Size measures (Berk 1995, 1997)
- Inefficient market proxies (Ferguson and Shockley 2003)

Besides: the "data snooping" issue

- Individual data mining
- Collective data mining
- Journal publication policy

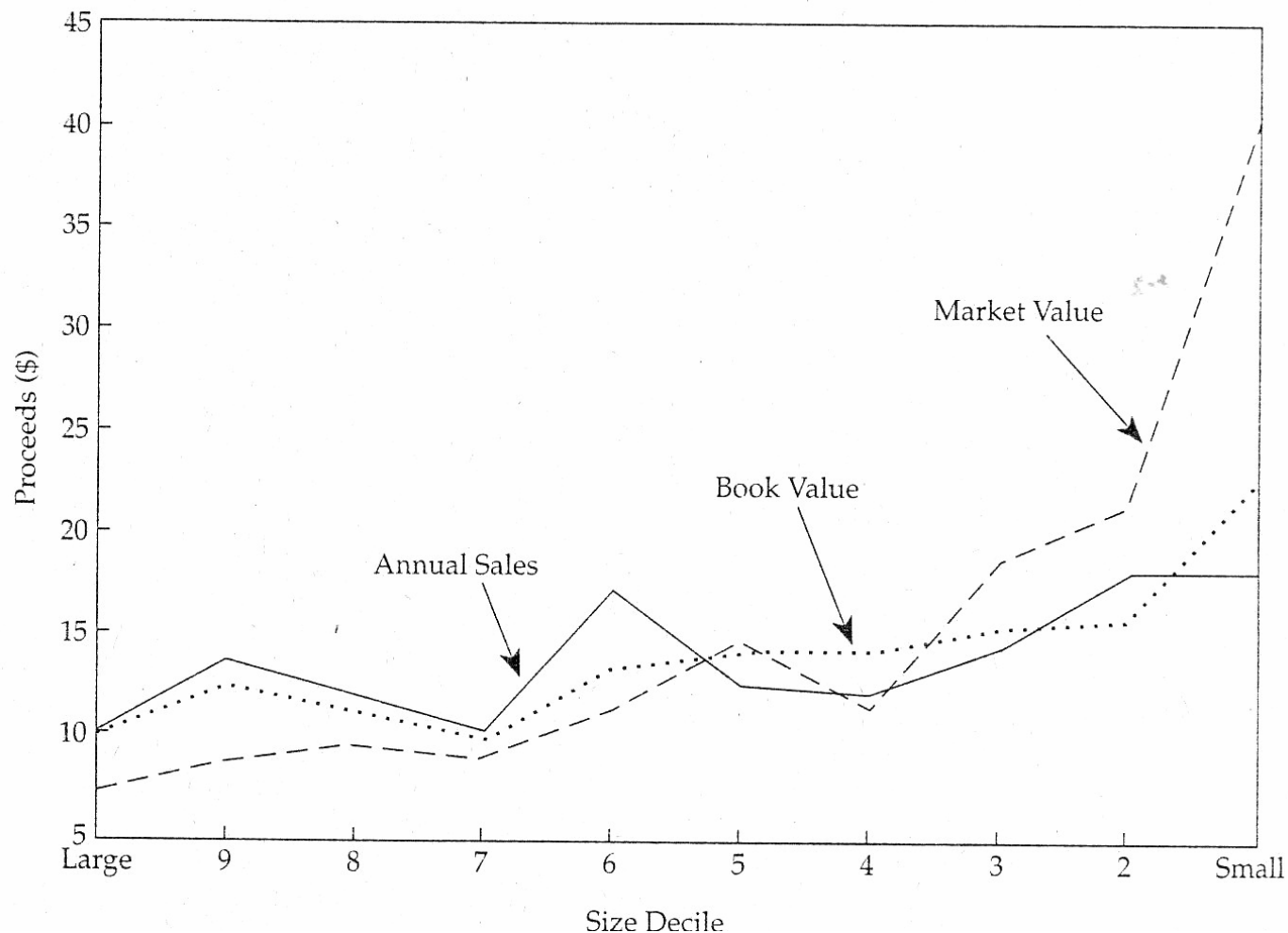
Remedies: Out-of-sample tests (but the availability of new samples is limited, and asset pricing tests require long time series)

Size Measures (Berk 1995, 1997)

- Size measure: market value of equity
- The market value measures
 - The expected value of future cash flows
 - The discount rate (which is related to risk)
- Assume two firms with identical cash flows but different risk
 - The riskier firm has lower market value
 - The riskier firm has higher expected return
 - We observe a size effect!
- Thus: Even if there is no true size effects, we will observe one because our size measure is related to the discount rate and, thus, to risk

- Other size measures are at best weakly related to returns (Berk 1997, Figure 1)

Figure 1. Proceeds of \$1 Invested from 1967 to 1987, by Size Decile and Size Variable Used in Sort



Inefficient Market Proxies (Ferguson and Shockley 2003)

- Betas are usually measured relative to equity-only market proxies
- This induces bias
- Let M , E , D be the total market value of the asset, the equity and the debt market. Then the covariance and true beta for stock i are:

$$\sigma_{i,M} = \frac{E}{M} \sigma_{i,E} + \frac{D}{M} \sigma_{i,D} ; \beta_i = \frac{\sigma_{i,M}}{\sigma_M^2} = \frac{E}{M} \frac{\sigma_{i,E}}{\sigma_M^2} + \frac{D}{M} \frac{\sigma_{i,D}}{\sigma_M^2}$$

- Beta measured relative to the equity market only is:

$$\hat{\beta}_i^E = \frac{\sigma_{i,E}}{\sigma_E^2} = \frac{M}{E} \frac{\sigma_M^2}{\sigma_E^2} \left[\beta_i - \frac{D}{M} \frac{\sigma_D^2}{\sigma_M^2} \hat{\beta}_i^D \right]$$

- The estimated beta is biased by
 - a scaling factor that is the same for all assets (and thus inconsequential in many applications)
 - an asset-specific term that is related to the debt market beta of the asset
- The debt market beta is likely to be related to leverage and default risk
- Thus: The measurement bias is systematically related to variables that proxy for leverage and default risk
- Size and BE/ME *do* proxy for leverage and default risk
- A 3-factor model with the market risk premium and returns on portfolios formed on relative leverage and relative distress (measured by Altman's Z-score) outperforms the FF93 model [method: generalized Fama and MacBeth, i.e., first-pass beta estimation for *all* factors]

Panel B: Results for Market, Leverage, and Distress Three-Factor Model with
Marginal Contribution of Size and Book-to-Market Factors,

$$RP_{i,t} - RF_t = \gamma_0 + \gamma_{MKT} \hat{\beta}_i^{MKT} + \gamma_{D/E} \hat{\beta}_i^{D/E} + \gamma_Z \hat{\beta}_i^Z + \gamma_{SMB \perp} \hat{\beta}_i^{SMB \perp} + \gamma_{HML \perp} \hat{\beta}_i^{HML \perp} + \varepsilon_{i,t}$$

	γ_0	γ_{MKT}	$\gamma_{D/E}$	γ_Z	$\gamma_{SMB \perp}$	$\gamma_{HML \perp}$	
Coefficient	0.74	-0.27	2.18	1.53	-0.35	0.17	
<i>t</i> -value	1.97	-0.63	3.11	3.35	-1.65	0.92	
<i>p</i> -value	4.92	53.20	0.20	0.09	10.01	36.03	$R^2 = 81\%$
Corrected <i>t</i>	1.97	-0.56	3.00	3.23	-1.34	0.82	
Corrected <i>p</i>	4.92	57.63	0.29	0.13	18.12	55.33	

(Ferguson and Shockley 2003, Table 4, Panel B)

- A time series test using the Gibbons et al. (1989) test of the null that the intercepts are jointly zero does not yield satisfactory results

Anomalies / Market Inefficiency

There is some evidence that is not easily reconciled with a risk-factor interpretation. Some examples:

- The size effect is mainly due to a January seasonal
 - It is difficult to find an explanation why small stocks should have higher returns only in January
 - The fact that FF93 find no January seasonal in their residuals does not invalidate the argument
- One should expect that the size and BE/ME factors are correlated across countries. They aren't (Hawawini and Keim 1998, Table 8; upper/right: size; lower/left: BE/ME, 1975-1994)

	France	UK	Germany	Japan	US
France		0.07	0.15	0.00	0.07
UK	0.14		0.02	0.12	0.24
Germany	0.13	0.20		0.09	0.07
Japan	0.00	0.17	0.09		-0.02
US	0.16	0.29	0.25	0.10	

- If size and book to market were priced risk factors, we should expect that the return covariation drives the return differences
 - stocks whose return exhibit large correlation with the size factor should have high returns *irrespective* of the size of the firm
 - Daniel and Titman (1997) find that it is the characteristic itself (and not the covariation pattern) that explains the return differences
 - "Once we control for firm characteristics, expected returns do not appear to be positively related to the loadings on the market, HML or SMB factors" (p. 4)
- Generally, test of equilibrium models are joint tests that a) the model is the correct equilibrium model and b) the market is informationally efficient (in the semi-strong form)
- Alternative explanations: Behavioral Finance

V.3. A Skeptical Appraisal (Lewellen et al. 2006)

The current practice:

- Dependent variables: Return on the 25 FF size / BE/ME portfolios
- Propose factors (e.g. labor income, GDP, housing prices, liquidity risk etc.)
- Regress the returns on their factor loadings and test whether the slopes are significant (i.e., whether the factors are priced)

Observations on the FF portfolios

- They have a strong factor structure
- The FF93 factors explain more than 90% of the time-series variation of the returns

Why is that a problem?

- Assume that the true returns \mathbf{R} have a perfect factor structure; let \mathbf{F} be the vector of factors and \mathbf{B} be the factor loadings
- The "true model" thus is

$$\mathbf{R} = \mathbf{B}\mathbf{F} + \mathbf{e}$$

$$\boldsymbol{\mu} = \mathbf{B}\boldsymbol{\mu}_F$$

- Let \mathbf{P} be the "proposed model" (with the same number of factors as \mathbf{F}) and let \mathbf{C} be the associated matrix of factor loadings.
- The model is tested by estimating

$$\boldsymbol{\mu} = \mathbf{z}\mathbf{1} + \mathbf{C}\boldsymbol{\lambda} + \boldsymbol{\eta}$$

and testing whether $\boldsymbol{\lambda} \neq \mathbf{0}$

where

$$\mathbf{C} = \frac{\text{Cov}(\mathbf{R}, \mathbf{P})}{\text{Var}(\mathbf{P})}$$

- Assume that P does not explain anything of the residual variation of the true model (i.e., $\text{Cov}(\mathbf{e}, \mathbf{P}) = 0$) but is correlated with \mathbf{F} , let the correlation matrix between \mathbf{F} and \mathbf{P} be non-singular
- Because of $\text{Cov}(\mathbf{e}, \mathbf{P}) = 0$ it must be that $\text{Cov}(\mathbf{R}, \mathbf{P}) = \mathbf{B}\text{Cov}(\mathbf{F}, \mathbf{P})$
- But then

$$\mathbf{C} = \mathbf{B} \frac{\text{Cov}(\mathbf{F}, \mathbf{P})}{\text{Var}(\mathbf{P})} = \mathbf{B}\mathbf{Q}; \quad \mathbf{Q} \equiv \frac{\text{Cov}(\mathbf{F}, \mathbf{P})}{\text{Var}(\mathbf{P})}$$

and

$$\boldsymbol{\mu} = \mathbf{B}\boldsymbol{\mu}_F = \mathbf{C}\mathbf{Q}^{-1}\boldsymbol{\mu}_F = \mathbf{C}\boldsymbol{\lambda}; \quad \boldsymbol{\lambda} \equiv \mathbf{Q}^{-1}\boldsymbol{\mu}_F$$

- Interpretation: Expected returns are linear in \mathbf{P} as long as \mathbf{P} has non-zero correlation with \mathbf{F}
- This implies that any factors that are correlated with size and BE/ME will appear to be priced

Implications:

- Given the strong factor structure of the FF portfolios it is not difficult to identify factors that apparently explain the returns and yield high R^2 s
- But then, finding that certain factors are priced is a rather weak result

What can be done?

- Use other "test assets" (e.g. industry portfolios)
- Take parameter restrictions seriously
 - often only $\lambda \neq \mathbf{0}$ is tested
 - The intercept should be close to the risk-free rate
 - slopes should be close to the average risk premium
- Use GLS rather than OLS

- Report confidence intervals rather than p-values.
 - They show all true parameter values that are consistent with the data
 - The problem of the "reversed null hypothesis" is avoided

Lewellen et al. (2006) apply their remedies to some recently proposed models and obtain disappointing results - none of the models works well

VI. Further Topics

Outline:

1. Liquidity and Expected Returns
2. Time-varying expected returns and conditional tests
(could be continued...)

VI.1. Expected Returns and Liquidity

Investors are interested in their net return:

- Transaction costs decrease net returns
- Investors willingness to pay should be net of the discounted transaction costs (for both buy and sale)

Consequences (Amihud / Mendelson 1986):

- Assets with higher transaction costs (higher spreads) should have higher (gross) returns
- This may (partially) explain the size effect!
- There will be a clientele effect: Investors with longer investment horizons will hold less liquid assets (and earn excess returns)

An Empirical approach (Datar et al. 1998):

- Fama / French with an additional liquidity factor
- Dependent variable: monthly stock returns
- Liquidity measure: Turnover ratio T

$$T = \frac{\text{Share trading volume}}{\text{Shares outstanding}}$$

- Control variables
 - Beta (obtained from time-series regression)
 - Size (log of market capitalization)
 - Book-to-market ratio
- Sample: NYSE, monthly data 1962-1991

Results:

Constant	Turnover	Book-to-market	Log of size	Beta
Panel A				
0.73 (32.51)	- 0.04 (- 8.86)			
0.83 (34.19)	- 0.04 (- 9.07)	0.21 (10.15)		
2.39 (12.32)	- 0.05 (- 10.58)		- 0.07 (- 7.48)	
1.69 (8.09)	- 0.05 (- 10.56)	0.16 (6.99)	- 0.04 (- 3.68)	
2.30 (9.70)	- 0.04 (- 8.58)	0.14 (5.97)	- 0.05 (- 4.65)	- 0.37 (- 5.76)

Source: Datar et al. (1998), 211

- Stocks with lower liquidity offer higher returns
- Stock market liquidity thus affects expected returns (and the cost of capital)
 - Managers and shareholders should be interested in increasing the liquidity of the stock
- The liquidity effect persists even after controlling for the "Fama-French factors"

Additional evidence:

- Several other papers have confirmed the existence of a relation between liquidity and returns
- Bekaert et al. (2005) analyze a sample from 19 emerging markets and also confirm the relation between liquidity and expected returns
- Measures that increase liquidity (i.e. transferring a stock to continuous trading) are associated with positive abnormal returns (Amihud / Mendelson / Lauterbach 1997)

Easley et al. (2002) argue that it is not liquidity *per se* that is related to expected returns but the risk of encountering an opponent with superior information

VI.2. Time-Varying Expected Returns and Conditional Tests

Introduction:

- The CAPM is a one-period model
- Tests using time-series data implicitly assume stability
- However, both betas and the expected market risk premium may change over time
- There is some evidence that market-level returns are predictable
- But then the variables explaining the expected market risk premium should be incorporated as conditioning variables in tests of asset pricing models

Return predictability:

- Stock returns (at the market level) appear to be predictable on the basis of variables such as:
 - dividend yields
 - price-earnings ratios
 - the yield curve (term and risk premium)

Example (inspired by Cochrane 1999, p. 44)

Data (from Germany):

- *Price-Dividend-Ratio* (dividends include tax credit from 1977 onwards) δ_t , December 1960-1994
- Annual DAFOX returns 1961-1999
- Riskless interest rate (3 months) 1961-1999

Regression:

$$r_{t,t+\tau}^{\text{DAFOX}} - r_{t,t+\tau}^f = \alpha + \beta \frac{1}{\delta_t} + \varepsilon_t$$

Results:

	β	t-value	R^2
1 year	-1,251	2,35	0,14
5 years	-4,886	4,02	0,33

- Long-term market level returns appear to be predictable

Implications:

- Time-varying betas may help to explain the size and book-to-market effect
- Example: If small firms have time-varying betas and the changes in beta are correlated with the (time-varying) market risk premium, then an unconditional model will identify a size effect
- In a conditional model, the CAPM could hold

"Theoretically, it is well known that the conditional CAPM could hold perfectly, period by period, even though stocks are mispriced by the unconditional CAPM" (Lewellen and Nagel 2006)

Does the conditional CAPM explain asset pricing "anomalies"?

- Lewellen and Nagel (2006) argue that deviations from the unconditional CAPM are too large
- The alpha (the pricing error relative to the unconditional model) of a stock depends on the covariance between its (time-varying) beta and the (time-varying) market risk premium
- Assuming plausible values for the standard deviations, $\rho_{(\tilde{r}_{m,t}-r_{f,t}; \beta_{i,t})} = 1$ puts an upper bound on the unconditional alpha
- Empirical alphas are much larger than the upper bound

Testing the Conditional CAPM

- Starting point is the unconditional Black / Jensen / Scholes test procedure:

$$\left(\mathbf{r}_{i,t} - \mathbf{r}_{f,t} \right) = \boldsymbol{\alpha}_i + \boldsymbol{\beta}_i \left(\mathbf{r}_{m,t} - \mathbf{r}_{f,t} \right)$$

$$H_0 : \boldsymbol{\alpha}_i = \mathbf{0}$$

- The betas are now modeled as functions of (macroeconomic) variables
- Applications
 - conditional tests of asset pricing models
 - conditional performance evaluation (e.g. Ferson and Schadt 1996)
- Problem: The procedure assumes that the researcher knows the set of conditioning variables

An alternative approach (Lewellen and Nagel 2006):

- Estimate

$$\left(\mathbf{r}_{i,t} - \mathbf{r}_{f,t} \right) = \alpha_i + \beta_i \left(\mathbf{r}_{m,t} - \mathbf{r}_{f,t} \right)$$

for short periods using data of higher frequency (e.g. one quarter with daily data)

- Dimson (1979) betas to account for infrequent trading
- Idea: beta should be constant over short periods of time
- Advantage: No need to specify the conditioning variables
- Two tests:
 - Is the time-series average of the conditional alphas equal to zero?
 - Does the time series of conditional betas covary with the market risk premium?

- Tests are performed on size, book-to-market and momentum portfolios
- If the conditional CAPM explains these "anomalies", than the time series average of conditional alphas should be zero

Results:

- Average conditional alphas for book-to-market and momentum portfolios remain large and significant (and similar in magnitude to the unconditional alphas):

	Size			B/M			Momentum		
	Small	Big	S-B	Grwth	Value	V-G	Losers	Winrs	W-L
<i>Average conditional alpha (%)</i>									
Quarterly	0.42	0.00	0.42	-0.01	0.49	0.50	-0.79	0.55	1.33
Semiannual 1	0.26	0.00	0.26	-0.08	0.40	0.47	-0.61	0.39	0.99
Semiannual 2	0.16	0.01	0.15	-0.12	0.36	0.48	-0.83	0.53	1.37
Annual	-0.06	0.08	-0.14	-0.20	0.32	0.53	-0.56	0.21	0.77

(Lewellen and Nagel 2006, Table 3); bold values are significant

- Estimated betas vary significantly with some conditioning variables (the T-bill rate, the dividend yield and the term premium):

Panel B: Betas regressed on the state variables^a

Slope estimate

β_{t-1}	0.12	0.05	0.11	0.10	0.12	0.08	0.10	0.15	0.22
$R_{M,-6}$	0.05	-0.01	0.04	0.02	0.04	0.04	-0.19	0.20	0.39
TBILL	-0.13	-0.02	-0.11	-0.03	-0.14	-0.13	0.09	-0.14	-0.24
DY	0.14	0.05	0.09	0.06	0.16	0.10	-0.07	0.11	0.19
TERM	-0.10	0.00	-0.10	-0.02	-0.08	-0.07	0.07	-0.11	-0.19
CAY	-0.05	0.02	-0.08	-0.03	-0.01	0.03	0.00	-0.01	-0.01

(Lewellen and Nagel 2006, Table 5); bold values are significant

- The covariance between conditional betas and the market risk premium (which is approximately equal to the implied unconditional alpha that one should observe if the conditional model is correct) is too small

Panel A: Covariance between estimated betas and market returns (% monthly)

Est.	Qtr	-0.32	0.07	-0.39	-0.20	-0.12	0.09	0.16	-0.23	-0.38
	Semi 1	-0.17	0.07	-0.24	-0.14	-0.03	0.11	-0.03	-0.07	-0.04
	Semi 2	-0.12	0.07	-0.19	-0.10	-0.03	0.07	0.15	-0.18	-0.33
	Annual	0.06	0.03	0.03	-0.03	0.01	0.04	-0.08	0.11	0.20

(Lewellen and Nagel 2006, Table 6); bold values are significant

Punchline: The conditional CAPM does not explain the size, book-to-market and momentum anomalies

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