

Advanced Mathematical Methods

WS 2018/19

1 Linear Algebra

PD Dr. Thomas Dimpfl

Chair of Statistics, Econometrics and Empirical Economics

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN



WIRTSCHAFTS- UND
SOZIALWISSENSCHAFTLICHE
FAKULTÄT

Outline: Linear Algebra

1.9 Quadratic forms and sign definiteness

Readings

- ▶ Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne Strøm. *Further Mathematics for Economic Analysis*. Prentice Hall, 2008 Chapter 1

Online Resources

MIT course on Linear Algebra (by Gilbert Strang)

- ▶ Lecture 26: Symmetric matrices and positive definiteness
<https://www.youtube.com/watch?v=umt6BB1nJ4w>
- ▶ Lecture 27: Positive definite matrices and minima – Quadratic forms
<https://www.youtube.com/watch?v=vF7eyJ2g3kU>

1.9 Quadratic forms and sign definiteness

Definitions

- ▶ Degree of a polynomial
- ▶ Form of n th degree
- ▶ special case: quadratic form

$$Q(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

1.9 Quadratic forms and sign definiteness

A quadratic form $Q(x_1, x_2)$ for two variables x_1 and x_2 is defined as

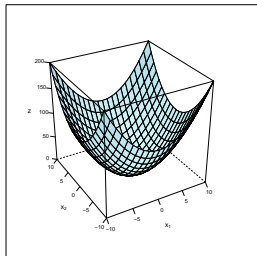
$$Q(x_1, x_2) = \underset{(1 \times 2)}{\mathbf{x}'} \underset{(2 \times 2)}{\mathbf{A}} \underset{(2 \times 1)}{\mathbf{x}} = \sum_{i=1}^2 \sum_{j=1}^2 a_{ij} x_i x_j$$

where $a_{ij} = a_{ji}$ and, thus,

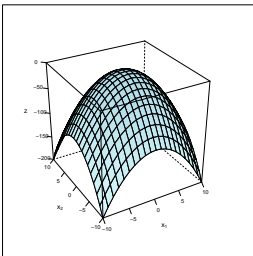
with the symmetric coefficient matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$

1.9 Quadratic forms and sign definiteness

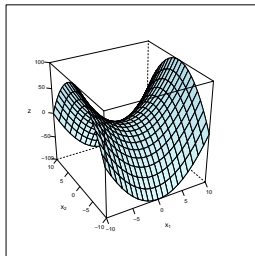
Graph of the positive definite form $Q(x_1, x_2) = x_1^2 + x_2^2$



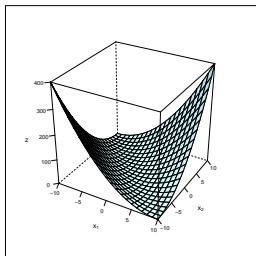
Graph of the negative definite form $Q(x_1, x_2) = -x_1^2 - x_2^2$



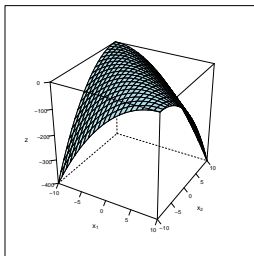
Graph of the indefinite form $Q(x_1, x_2) = x_1^2 - x_2^2$



Graph of the positive semidefinite form $Q(x_1, x_2) = (x_1 + x_2)^2$



Graph of the negative semidefinite form $Q(x_1, x_2) = -(x_1 + x_2)^2$



1.9 Quadratic forms and sign definiteness

The quadratic form associated with the matrix \mathbf{A} (and thus the matrix \mathbf{A} itself) is said to be

positive definite,	if $Q = \mathbf{x}'\mathbf{A}\mathbf{x} > 0$	for all $\mathbf{x} \neq \mathbf{0}$
positive semi-definite,	if $Q = \mathbf{x}'\mathbf{A}\mathbf{x} \geq 0$	for all \mathbf{x}
negative definite,	if $Q = \mathbf{x}'\mathbf{A}\mathbf{x} < 0$	for all $\mathbf{x} \neq \mathbf{0}$
negative semi-definite,	if $Q = \mathbf{x}'\mathbf{A}\mathbf{x} \leq 0$	for all \mathbf{x}

Otherwise the quadratic form is **indefinite**.

Note: For any quadratic matrix \mathbf{A} it holds that $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{B}\mathbf{x}$ with $\mathbf{B} = 0,5 \cdot (\mathbf{A} + \mathbf{A}')$ symmetric.

1.9 Quadratic forms and sign definiteness

The quadratic form $Q(\mathbf{x})$ is

- ▶ positive (negative) definite, if **all** eigenvalues of the matrix \mathbf{A} are positive (negative): $\lambda_j > 0$ ($\lambda_j < 0$) $\forall j = 1, 2, \dots, n$;
- ▶ positive (negative) semi-definite, if **all** eigenvalues of the matrix \mathbf{A} are non-negative (non-positive): $\lambda_j \geq 0$ ($\lambda_j \leq 0$) $\forall j = 1, 2, \dots, n$ and **at least one** eigenvalue is equal to zero;
- ▶ indefinite, if two eigenvalues have different signs.

1.9 Quadratic forms and sign definiteness

Properties of positive definite and positive semi-definite matrices

- 1) Diagonal elements of a positive definite matrix are strictly positive. Diagonal elements of a positive semi-definite matrix are nonnegative.
- 2) If \mathbf{A} is positive definite, then \mathbf{A}^{-1} exists and is positive definite.
- 3) If \mathbf{X} is $n \times k$, then $\mathbf{X}'\mathbf{X}$ and $\mathbf{X}\mathbf{X}'$ are positive semi-definite.
- 4) If \mathbf{X} is $n \times k$ and $\text{rk}(\mathbf{X}) = k$, then $\mathbf{X}'\mathbf{X}$ is positive definite (and therefore non-singular).