Advanced Mathematical Methods

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1 Linear Algebra

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Outline: Linear Algebra

Readings

- Knut Sydsaeter and Peter Hammond. *Essential Mathematics for Economic Analysis*.
  Prentice Hall, third edition, 2008 Chapters 15-16
- Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne

  Strøm. *Further Mathematics for Economic Analysis*.
  Prentice Hall, 2008 Chapter 1

Online Resources

MIT course on Linear Algebra (by Gilbert Strang)

- Lecture 1: Vectors, Matrices
  https://www.youtube.com/watch?v=ZK3O402wf1c
- Lecture 3: Multiplication and Inverse Matrices
  https://www.youtube.com/watch?v=QVKj3LADCnA
- Lecture 9: Independence, basis and dimension
  https://www.youtube.com/watch?v=yjBerM5jWsc
- Lecture 18: Properties of determinants
  https://www.youtube.com/watch?v=srxexLishgY

1.1 Vectors

Vector operations

multiplication of an n-dimensional vector $\boldsymbol{v}$ with a scalar $c \in \mathbb{R}$:

$$c \cdot \underset{(n\times 1)}{\boldsymbol{v}} = \begin{pmatrix} c \cdot v_1 \\ \vdots \\ c \cdot v_n \end{pmatrix}$$

sum of two $n-$dimensional vectors $\boldsymbol{v}$ und $\boldsymbol{w}$:

$$\underset{(n\times 1)}{\boldsymbol{v}} + \underset{(n\times 1)}{\boldsymbol{w}} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

The **difference** between two $n-$dimensional Vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is obtained by $\boldsymbol{v} - \boldsymbol{w} = \boldsymbol{v} + (-1)\boldsymbol{w}$.

1.1 Vectors

Vector operations

Inner product (Scalar product) $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$:

$$\underset{(1\times n)}{\boldsymbol{v}'}\underset{(n\times 1)}{\boldsymbol{w}} = \underset{(1\times 1)}{\sum_{i=1}^{n} v_i w_i}$$

Orthogonality of two vectors: $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$:

$$\underset{(1\times n)}{\boldsymbol{v}'}\underset{(n\times 1)}{\boldsymbol{w}} = \underset{(1\times 1)}{\sum_{i=1}^{n} v_i w_i} = 0$$

1.2 Matrices

Matrix operations

Multiplication with a scalar:

$$\boldsymbol{C} = k \cdot \boldsymbol{A} \quad \Leftrightarrow \quad c_{ij} = k \cdot a_{ij} \quad \forall \quad i, j.$$

Addition (Subtraction) of matrices:

for two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ with the same dimensions

$$\boldsymbol{C} = \boldsymbol{A} \pm \boldsymbol{B} \;\Leftrightarrow\; c_{ij} = a_{ij} \pm b_{ij} \qquad \forall\; i, j.$$

1.2 Matrices

Matrix multiplication

$$\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B}$$

with

$$c_{kl} = \sum_{i=1}^{m} a_{ki} \cdot b_{il}$$

Note: Conformity and dimensionality.

$$\underset{(n \times p)}{\boldsymbol{C}} = \underbrace{\underset{(n \times \underbrace{m)}{\boldsymbol{A}} \cdot \underset{(m}_{\text{conformity}} \times p)}{\boldsymbol{B}}}_{\text{dimensionality}}$$

1.2 Matrices

Rules of matrix multiplication

Given conformity, it holds that:

- $(\boldsymbol{A} \cdot \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot (\boldsymbol{B} \cdot \boldsymbol{C})$ (associative law)
- $(\boldsymbol{A} + \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{C}$ (distributive law from the right)
- $\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$ (distributive law from the left)

Power of a matrix: For a quadratic matrix $\boldsymbol{A}$ we calculate the non-negative integer power as follows:

$$\boldsymbol{A}^n = \underbrace{\boldsymbol{A}\boldsymbol{A}\cdots\boldsymbol{A}}_{n\text{times}} \qquad \text{with} \quad n > 0$$

special case: $\boldsymbol{A}^0 = \boldsymbol{I}$.

1.2 Matrices

Kronecker product

$\boldsymbol{A}$ is $m \times n$ and $\boldsymbol{B}$ is $p \times q$, then the Kronecker product $\boldsymbol{A} \otimes \boldsymbol{B}$ is the $mp \times nq$ block matrix

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & \dots & a_{1n}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & \dots & a_{2n}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\boldsymbol{B} & \dots & a_{mn}\boldsymbol{B} \end{bmatrix}$$

1.2 Matrices

Idempotent matrix:
A quadratic matrix $\boldsymbol{A}$ is idempotent if: $\boldsymbol{A}^2 \equiv \boldsymbol{A}\boldsymbol{A} = \boldsymbol{A}$.

Trace of a quadratic matrix:

$$\boldsymbol{tr}(\boldsymbol{A}) \equiv \sum_{i=1}^{n} a_{ii}$$

1.3 Inverse of a quadratic matrix

The inverse of a matrix $\boldsymbol{A}$, expressed by $\boldsymbol{A}^{-1}$, has the following characteristics:

$$\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I}$$

Note:

1.) The matrix $\boldsymbol{A}$ has to be quadratic (due to conformity). Otherwise it is not invertible.

2.) The inverse doesn't have to exist for every single quadratic matrix

3.) If there is an inverse, we call the quadratic matrix *non-singular* or *regular*, otherwise we call it *singular*.

1.3 Inverse of a quadratic matrix

4.) If there is an inverse, then it is unambiguous

<u>Characteristics</u> (for non-singular matrices $\boldsymbol{A}, \boldsymbol{B}$):

- $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
- $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
- $(\boldsymbol{A}')^{-1} = (\boldsymbol{A}^{-1})'$

1.4 The determinant

What is a determinant? Some intuition and why it is important!

The determinant ...

... is a single number that contains information about a square matrix $\boldsymbol{A}$.

... tells us whether the matrix $\boldsymbol{A}$ is singular.

... turns up in most formulas in linear algebra, e.g. for the calculation of inverses or the determination of the rank of the matrix.

... is informative w.r.t. eigenvalues and whether the matrix can be positive, negative or indefinite.

1.4 The determinant

How to calculate the determinant - Sarrus' Rule

For a 2×2 matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the determinant is defined as follows:

$$\det(\boldsymbol{A}) = |\,\boldsymbol{A}\,| = a_{11}\,a_{22} - a_{12}\,a_{21}$$

1.4 The determinant

An important application:

In general we can show that the determinant of a quadratic matrix with **linearly dependent columns (or rows)** has a zero determinant.

$\Longrightarrow$ The determinant criterion gives us information about the linear dependency (or independency) of the rows (or rather columns) of a matrix as well as about the existence of its inverse.

$\Longrightarrow$ If $\det(\boldsymbol{A}) = 0$ the matrix is singular, whereas if $\det(\boldsymbol{A}) \neq 0$ it is invertible!

1.4 The determinant

How to calculate the determinant - Cofactor expansion

Calculation of the determinant for general $n \times n$ matrices:
Cofactor expansion *across a row i:*

$$\det(\boldsymbol{A}) \;=\; \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij} \mid \boldsymbol{A}_{ij} \mid$$

Alternatively: Cofactor expansion *down a column j:*

$$\det(\boldsymbol{A}) \;=\; \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \mid \boldsymbol{A}_{ij} \mid$$

Note: The product $(-1)^{i+j} \mid \boldsymbol{A}_{ij} \mid$ is called **cofactor** and $\boldsymbol{A}_{ij}$ is the minor.

1.4 The determinant

The determinant of the (3×3)-matrix $\boldsymbol{A}$ is defined as

$$\det(\boldsymbol{A}) \;=\; a_{11} \cdot \mid \boldsymbol{A}_{11} \mid - \, a_{12} \cdot \mid \boldsymbol{A}_{12} \mid + \, a_{13} \cdot \mid \boldsymbol{A}_{13} \mid$$

(cofactor formula)

1.4 The determinant

Illustration:

$$\underset{(3\times 3)}{\boldsymbol{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Determining the **submatrices**:

Elimination of the 1^{st} row and the 1^{st} column of $\boldsymbol{A}$ yields the **submatrix** $\boldsymbol{A}_{11}$ of dimension (2×2):

$$\underset{(3\times 3)}{\boldsymbol{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \Longrightarrow \quad \underset{(2\times 2)}{\boldsymbol{A}_{11}} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

1.4 The determinant

Elimination of the 1^{st} row and the 2^{nd} column of $\boldsymbol{A}$ yields the **submatrix** $\boldsymbol{A}_{12}$ of dimension (2×2):

$$\underset{(3\times3)}{\boldsymbol{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies \underset{(2\times2)}{\boldsymbol{A}_{12}} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

Elimination of the 1^{st} row and the 3^{rd} column of $\boldsymbol{A}$ yields the **submatrix** $\boldsymbol{A}_{13}$ of dimension (2×2):

$$\underset{(3\times3)}{\boldsymbol{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \implies \underset{(2\times2)}{\boldsymbol{A}_{13}} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

The determinants $|\boldsymbol{A}_{ij}|$ of the submatrices $\boldsymbol{A}_{ij}$ are called **subdeterminants**; They can be calculated using the *Sarrus' Rule* (if of order of 3 or lower)

1.4 The determinant

How to calculate the determinant - Sarrus' Rule revisited

Extension of the 3×3 matrix $\boldsymbol{A}$ for the application of the *Sarrus' Rule*:

$$\boldsymbol{A}^\star = \left(\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right)$$

$$\begin{aligned} \det(\boldsymbol{A}) \;=\; & a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} \\ & - a_{13}\,a_{22}\,a_{31} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} \end{aligned}$$

1.4 The determinant

Properties of determinants

for $\boldsymbol{A}$ and $\boldsymbol{B}$ with dimension $n \times n$:

1.) The exchange of two rows or two columns of a matrix leads to a change in the sign of the determinant.

2.) The determinant doesn't change its value if we add the multiple of a row (column) to another row (column) within a matrix. **Elimination does not change the determinant**.

3.) The determinants of a matrix and its transpose are equal:

$$\det(\boldsymbol{A}) = \det(\boldsymbol{A}')$$

4.) Multiplying all components of a $n \times n$ matrix with the same factor k leads to a change in the value of the determinant by the factor k^n: **Determinant is linear in each row.**

$$\det(k\boldsymbol{A}) = k^n \det(\boldsymbol{A})$$

1.4 The determinant

Properties of determinants

5.) The determinant of every identity matrix is equal to 1; the determinant of every zero matrix is equal to 0.

6.) The determinant of the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ equals the product of the determinants of $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\det(\boldsymbol{A} \cdot \boldsymbol{B}) = \det(\boldsymbol{A}) \cdot \det(\boldsymbol{B})$$

7.) From 6.) follows for a regular matrix $\boldsymbol{A}$ that:

$$\det(\boldsymbol{A}^{-1}) = \frac{1}{\det(\boldsymbol{A})}$$

8.) In general: $\det(\boldsymbol{A} + \boldsymbol{B}) \neq \det(\boldsymbol{A}) + \det(\boldsymbol{B})$.

1.4 The determinant

Properties of determinants

9.) If $\det(\boldsymbol{A}) = 0$ the matrix has linearly dependent rows (columns) and is singular.

10.) The determinant of an upper (lower) triangular matrix $n \times n$ matrix $\boldsymbol{U}$ is given by the product of the diagonal entries:

$$\det(\boldsymbol{U}) = \prod_{i=1}^{n} d_i$$

11.) The determinant of a diagonal matrix $n \times n$ matrix $\boldsymbol{D}$ is given by the product of the diagonal entries:

$$\det(\boldsymbol{D}) = \prod_{i=1}^{n} d_i$$

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1.5 Calculation of the inverse

We can determine regularity/non-singularity/invertibility of the *square matrix* $\boldsymbol{A}$ using the determinant. It holds that

$$\det(\boldsymbol{A}) \neq 0 \;\Leftrightarrow\; \boldsymbol{A}^{-1} \text{ exists.}$$

1.5 Calculation of the inverse

In general: The inverse of the $n \times n$ matrix $\boldsymbol{A}$ is denoted as

$$\boldsymbol{A}^{-1} = \boldsymbol{B} = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix}.$$

We get every single element of $\boldsymbol{B}$ by

$$b_{ij} = \frac{1}{|\boldsymbol{A}|}(-1)^{(i+j)} \; |\boldsymbol{A}_{ji}| . \qquad \text{(note the index!)}$$

In order to get the element b_{ij}, you have to calculate the subdeterminant $\boldsymbol{A}_{ji}$ crossing out the $j-$th row and the $i-$th column of $\boldsymbol{A}$.

1.6 Linear independence and rank of a matrix

Linear combination of vectors

Definition: linear combination

For the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ a n-dimensional vector $\boldsymbol{w}$ is called **linear combination** of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$, if there are real numbers $c_1, c_2, \ldots, c_k \in \mathbb{R}$, such that:

$$\boldsymbol{w} = c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 + \cdots + c_k \cdot \boldsymbol{v}_k = \sum_{i=1}^{k} c_i \cdot \boldsymbol{v}_i \ .$$

1.6 Linear independence and rank of a matrix

Linear independence

Definition: linear independence

The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ are called **linearly independent**, if

$$c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 + \cdots + c_k \cdot \boldsymbol{v}_k = 0 \qquad \text{with} \quad c_1, c_2, \ldots, c_k \in \mathbb{R}$$

is only attainable with $c_1 = c_2 = \cdots = c_k = 0$. Otherwise they are called **linearly dependent** and $\boldsymbol{v}_1 = d_2 \cdot \boldsymbol{v}_2 + \cdots + d_k \cdot \boldsymbol{v}_k$ (with $d_2, d_3, \ldots, d_k \in \mathbb{R}$) applies.

1.6 Linear independence and rank of a matrix

The rank of a matrix

The **rank** of the $n \times m$ matrix $\boldsymbol{A}$ is determined by the maximum number of linearly independent columns (rows) of the matrix $\boldsymbol{A}$.

$$\boxed{\mathrm{rk}(\boldsymbol{A}) \leq \min(m, n)}$$

For every matrix the column rank equals the row rank.
The rank criterion allows to determine whether a quadratic $n \times n$ matrix $\boldsymbol{A}$ is regular/non-singular or not:

$$\begin{aligned} \mathrm{rk}(\boldsymbol{A}) = n \quad &\Rightarrow \quad non - singular \\ \mathrm{rk}(\boldsymbol{A}) < n \quad &\Rightarrow \quad singular \end{aligned}$$

1.6 Linear independence and rank of a matrix

Properties of the rank

1.) The rank of a matrix doesn't change if you exchange rows or columns among themselves.

2.) The rank of a matrix $\boldsymbol{A}$ is equal to the rank of the transpose $\boldsymbol{A}'$: $\text{rk}(\boldsymbol{A}) = \text{rk}(\boldsymbol{A}')$

3.) For a $m \times n$ matrix $\boldsymbol{A}$ the following applies: $\text{rk}(\boldsymbol{A}) = \text{rk}(\boldsymbol{A}'\boldsymbol{A})$, where $\boldsymbol{A}'\boldsymbol{A}$ is quadratic.

1.6 Linear independence and rank of a matrix

Determination of the rank of a matrix

1.) Consider all quadratic submatrices of a matrix of which the determinants are not 0. Then search for the submatrix with the highest order whose determinant is nonzero. The rank of the matrix is equal to the number of rows of this submatrix.

2.) Gaussian algorithm

3.) Eigenvalues

1.7 Linear equation systems

Linear combinations in matrix notation

Rewrite $\sum_{i=1}^{k} c_i \cdot \boldsymbol{v}_i = \boldsymbol{w}$ as

$$\underbrace{\begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_k \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}}_{\boldsymbol{x}} = \underbrace{\boldsymbol{w}}_{\boldsymbol{b}}$$

where $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ and

- $\boldsymbol{A}$ is an $n \times k$ dimensional matrix
- $\boldsymbol{x}$ is an $k \times 1$ dimensional vector
- $\boldsymbol{b}$ is an $n \times 1$ dimensional vector.

1.7 Linear equation systems

How to solve a linear equation system

1. If $n > k$, i.e. there are more equations than unknowns, then there are infinitely many solutions to the equation.
2. If $n < k$, i.e. there are fewer equations than unknowns, the system cannot be solved.
3. If $n = k$, $\boldsymbol{A} \cdot \boldsymbol{x} = 0$ is called a homogenous linear equation system. The equation system has a solution in any case. If $\boldsymbol{A}$ is singular, i.e. $\det(\boldsymbol{A}) = 0$, it has non-trivial solutions (infinitely many). If $\boldsymbol{A}$ is invertible, it has the trivial solution $\boldsymbol{x} = 0$.
4. If $n = k$, i.e. there are as many equations as unknowns, and the matrix $\boldsymbol{A}$ is invertible ($\text{rk}(\boldsymbol{A}) = n$ and $\det(\boldsymbol{A}) \neq 0$), then there exists a unique solution!

1.7 Linear equation systems

How to solve a linear equation system

For $n = k$ and $\det(\boldsymbol{A}) \neq 0$, three solution methods exist

1. solve $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ by Gaussian elimination
2. use the inverse $\boldsymbol{A}^{-1}$ to solve $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$
3. use Cramer's rule to get each element x_j in the vector $\boldsymbol{x}$:

$$x_j = \frac{|\boldsymbol{A}(j)|}{|\boldsymbol{A}|}$$

where in $A(j)$, the j^{th} column of $\boldsymbol{A}$ is replaced by $\boldsymbol{b}$.