Advanced Mathematical Methods

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1 Linear Algebra

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Outline: Linear Algebra

Readings

- Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne Strøm. *Further Mathematics for Economic Analysis.* Prentice Hall, 2008 Chapter 1

Online Resources

MIT course on Linear Algebra (by Gilbert Strang)

- Lecture 21: Eigenvalues and Eigenvectors
  https://www.youtube.com/watch?v=lXNXrLcoerU
- Lecture 22: Powers of a square matrix and Diagonalization
  https://www.youtube.com/watch?v=13r9QY6cmjc

1.8 Eigenvalues and eigenvectors

assume a scalar λ exists such that

$$\boldsymbol{Ax} = \lambda \boldsymbol{x}$$

λ: eigenvalue
$\boldsymbol{x}$: eigenvector

find λ via the homogenous linear equation system
$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = 0$

1.8 Eigenvalues and eigenvectors

The properties of a quadratic homogenous linear equation system imply that:

- in any case a solution does exist;
- if $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) \neq 0$, then $\bar{\boldsymbol{x}} = \mathbf{0}$ is the trivial solution;
- only if $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$ there is a non-trivial solution.

1.8 Eigenvalues and eigenvectors

Determination of the eigenvalues via *characteristic equation*:

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| \; = \; 0 \quad \Longleftrightarrow \quad (-1)^n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \ldots + \alpha_1 \lambda + \alpha_0 \; = \; 0$$

for every (real or complex) eigenvalue λ_i of the $(n \times n)$-Matrix $\boldsymbol{A}$ we can calculate the respective eigenvector $\boldsymbol{x}_i \neq \mathbf{0}$ solving the homogenous linear equation system

$$(\boldsymbol{A} - \lambda_i \boldsymbol{I}) \boldsymbol{x}_i = \mathbf{0} \; . \tag{1}$$

The properties of homogenous linear equation systems imply that the solution of eq. (1) is not unambiguous, i.e. for the eigenvalue λ_i we can find infinitely many eigenvectors $\boldsymbol{x}_i$

1.8 Eigenvalues and eigenvectors

$\boldsymbol{A}$ und $\boldsymbol{B}$ (quadratic matrices of order n) are similar if a regular $(n \times n)$ - matrix $\boldsymbol{C}$ exists, such that

$$\boldsymbol{B} = \boldsymbol{C}^{-1}\boldsymbol{A}\,\boldsymbol{C}\ .$$

Special case: **symmetric matrices**
For a symmetric $(n \times n)$-matrix $\boldsymbol{A}$ it holds that the normalized eigenvectors $\tilde{\boldsymbol{x}}_j$ with $j = 1, \ldots, n$ have the property

(1) $\tilde{\boldsymbol{x}}'_j\tilde{\boldsymbol{x}}_j = 1$ for all j and

(2) $\tilde{\boldsymbol{x}}'_i\tilde{\boldsymbol{x}}_j = 0$ for all $i \neq j$.

1.8 Eigenvalues and eigenvectors

Principle axis theorem

collecting the normalized eigenvectors $\tilde{\boldsymbol{x}}_j$ $(j = 1, \ldots, n)$ in a new matrix $\boldsymbol{T} = [\tilde{\boldsymbol{x}}_1 \cdots \tilde{\boldsymbol{x}}_n]$ with the property $\boldsymbol{T}^{-1} = \boldsymbol{T}'$ yields the diagonalization of $\boldsymbol{A}$ as follows:

$$\boldsymbol{D} = \boldsymbol{T}'\boldsymbol{A}\boldsymbol{T} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix}$$

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

1) The product of the eigenvalues of a $(n \times n)$-matrix yields its determinant: $|\boldsymbol{A}| = \prod_{i=1}^{n} \lambda_i$.
2) From 1.) it follows that a singular matrix must have at least one eigenvalue $\lambda_i = 0$.
3) The matrices $\boldsymbol{A}$ and $\boldsymbol{A}'$ have the same eigenvalues.
4) For a non-singular matrix $\boldsymbol{A}$ with eigenvalues λ we have: $|\boldsymbol{A}^{-1} - \frac{1}{\lambda} \boldsymbol{I}| = 0$.
5) Symmetric matrices have only real eigenvalues.
6) The rank of a symmetric matrix $\boldsymbol{A}$ is equal to the number of eigenvalues different from zero.