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Sequent Calculi and Bidirectional Natural Deduction: On the Proper Basis of Proof-theoretic Semantics

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Philosophical theories of logical reasoning are intrinsically related to formal models. This holds in particular of Dummett–Prawitz-style proof-theoretic semantics and calculi of natural deduction. Basic philosophical ideas of this semantic approach have a counterpart in the theory of natural deduction. For example, the "fundamental assumption" in Dummett's theory of meaning (Dummett, 1991, p. 254 and Ch. 12) corresponds to Prawitz's formal result that every closed derivation can be transformed into introduction form (Prawitz, 1965, p. 53). Examples from other areas in the philosophy of logic support this claim.

If conceptual considerations are genetically dependent on formal ones, we may ask whether the formal model chosen is appropriate to the intended conceptual application, and, if this is not the case, whether an inappropriate choice of a formal model motivated the wrong conceptual conclusions. We will pose this question with respect to the paradigm of natural deduction and proof-theoretic semantics, and plead for Gentzen's sequent calculus as a more adequate formal model of hypothetical reasoning. Our main argument is that the sequent calculus, when philosophically re-interpreted, does more justice to the notion of *assumption* than does natural deduction. This is particularly important when it is extended to a wider field of reasoning than just that based on logical constants.

To avoid confusion, a terminological *caveat* must be put in place: When we talk of the sequent calculus and the reasoning paradigm it represents, we mean, as its characteristic feature, its symmetry or bidirectionality, i.e.,

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the fact that it uses introduction rules for formulas occurring in different positions. We do not assume that these positions are syntactically represented by the left and right sides of a sequent, i.e., we do not stick to the *sequent format* which gave the calculus its name. In particular, we propose a natural-deduction variant of the sequent calculus called *bidirectional natural deduction*, which embodies the basic conceptual features of the sequent calculus.[1] Conversely, the natural-deduction paradigm to be criticized is the reasoning based on (conventional) introduction and elimination inferences, even though it can be given a sequent-calculus format as in so-called "sequent-style natural deduction".[2] The conceptual meaning of *natural deduction vs. sequent calculus*, which we try to capture by the notions of *unidirectionality vs. bidirectionality*, is to be distinguished from the particular syntax of these systems. We hope it will always be clear from the context whether a conceptual model or a specific syntactic format is meant.

We do not claim originality for the translation of the sequent calculus into bidirectional natural deduction. This translation is spelled out in detail in (von Plato, 2001). The system itself has been known much longer.[3] Here we want to make a philosophical point concerning the proper concept of hypothetical reasoning that also pertains to applications beyond logic and logical constants. The term "bidirectional natural deduction" seems to us to be a very appropriate characterization of the system considered. To our knowledge, it has not been used before.[4]

1 Assumptions in natural deduction

In a natural-deduction framework, there are essentially two things that can be done with assumptions: introducing and discharging. If we introduce an assumption

$$\begin{array}{c} A \\ \vdots \end{array}$$

[1] Other variants would be Schütte-style systems with metalinguistically specified right and left parts of formulas (Schütte, 1960) or even Frege-style systems, see (Schroeder-Heister, 1999).

[2] First suggested by Gentzen in (Gentzen, 1935), though not under that name.

[3] See, for example, (Tennant, 1992), (Tennant, 2002). For a brief history see (Schroeder-Heister, 2004, p. 33) (footnote). Unfortunately, the earliest proposal of this system (Dyckhoff, 1988) was accidentally omitted there.

[4] Von Plato (2001) simply speaks of "natural deduction with general elimination rules", which can also be understood in the unidirectional way (depending on the treatment of major premisses of elimination inferences). — The term "bidirectional" came up in personal discussions with Luca Tranchini on the proper treatment of negation in proof-theoretic semantics, a topic which is closely related to bidirectional reasoning. See his contribution to this volume (Tranchini, 2009).

then we make the derivation below A dependent on that assumption, and if we discharge it at an application of an inference

$$\begin{array}{c} {\scriptstyle (n)} \\ A \\ \vdots \\ \dfrac{B}{C}\,(n) \end{array}$$

we retract this dependency, so that the conclusion of that inference is not longer dependent on A. As proposed in (Prawitz, 1965), the numeral n indicates the link between assumptions and inferences at which they are discharged. Other notations are Fitch's (Fitch, 1952) explicit notation of subproofs, which goes back to (Jaśkowski, 1934), where the idea of discharging assumptions was developed even before (Gentzen, 1934/35).

Introducing and discharging assumptions is not very much one can do. Especially, there are no operations that change the form of an assumption and therefore have to do with its meaning. In this sense, they are purely structural operations. However, it is definitely more than can be done in Hilbert-type calculi, where we have at best the introduction of assumptions but never their discharging. In Hilbert-type systems assumptions can never disappear by means of a formal step. However, we can metalinguistically prove that we can work without assumptions by using them as the left side of a conditional statement. This is the content of the deduction theorem: If, in a Hilbert-type system, we have derived B from A, we can instead derive $A \to B$ by an appropriate transformation of the derivation of B from A.

Since in natural deduction we have the discharging of assumptions as a formal operation at the object level, we can express the content of the deduction theorem as a formal rule of implication introduction:

$$\begin{array}{c} {\scriptstyle (n)} \\ A \\ \vdots \\ \dfrac{B}{A \to B}\,(n) \end{array}$$

Although this is an important step beyond Hilbert-type calculi, it is not all that can possibly be done in extending the expressive power of formal systems. Our claim is that a genuinely semantic treatment of assumptions is more appropriate than a purely structural one as in natural deduction.

In natural deduction, assumptions have a close affinity to free variables: Assumptions which are not discharged are called *open*, whereas discharged assumptions are called *closed*. This terminology is justified since undischarged assumptions are open for the substitution of derivations whose end formula is the assumption in question, whereas closed assumptions are not.

Given a derivation $\begin{matrix} A \\ \mathcal{D} \\ B \end{matrix}$ with the open assumption A and a derivation $\begin{matrix} \mathcal{D}_1 \\ A \end{matrix}$ of A, then

$$\begin{matrix} \mathcal{D}_1 \\ A \\ \mathcal{D} \\ B \end{matrix}$$

is a derivation of B which may be considered a substitution instance of the original derivation. In this sense an open derivation corresponds to an open term, and a closed derivation, i.e. a derivation without open assumptions, corresponds to a closed term. This relationship between open and closed proofs and open and closed terms can be made formally explicit by a Curry–Howard-style association between terms and proofs, where the discharging of assumptions becomes a formal binding operation.

In our example, the derivation $\begin{matrix} \mathcal{D}_1 \\ A \end{matrix}$ can itself be open, just like a variable which is substituted with an open term. So in the formal concept of natural deduction and the composition of derivations there is no primacy of closed derivations over open ones. However, this primacy enters with the philosophical interpretation of natural deduction in the tradition of Dummett and Prawitz. There open assumptions are interpreted as placeholders for closed proofs.[5]

2 Assumptions in Dummett-Prawitz-style proof-theoretic semantics

Proof-theoretic semantics as advanced by Dummett and Prawitz[6] was framed by Prawitz in the form of a definition of validity of proofs, where a proof corresponds to a derivation in natural-deduction form. According to this definition, closed proofs in introduction form are primary as based on "self-justifying" steps, whereas the validity of closed proofs not in introduction form as well as the validity of open proofs is reduced to that of closed proofs using certain transformation procedures on proofs, called "justifications". Given a notion of validity for atomic proofs (i.e. proofs of atomic sentences), the definition of validity for the case of conjunction and implication formulas (to take two elementary cases) can be sketched as follows:

[5] Here we switch terminology from "derivation" to "proof", as in the semantical interpretation we are no longer dealing with purely formal objects, for which we reserve the term "derivation". Prawitz himself often speaks of "arguments" to avoid formalistic connotations still present with "proof".

[6] For an overview of this sort of semantics see (Schroeder-Heister, 2006) and the references therein.

- A closed proof of an atomic formula A is valid if there is a valid atomic proof of A.

- A closed proof of $A \wedge B$ in the introduction form

$$\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2 \\ A & B\end{matrix}}{A \wedge B}$$

  is valid if the subproofs $\mathcal{D}_1$ and $\mathcal{D}_2$ are valid closed proofs of A and B, respectively.

- A closed proof of $A \to B$ in the introduction form

$$\dfrac{\begin{matrix}(n) \\ A \\ \mathcal{D} \\ B\end{matrix}}{\mathrm{A \to B}}\,(n)$$

  is valid if for every closed proof $\begin{matrix}\mathcal{D}_1 \\ A\end{matrix}$ of A, the closed proof

$$\begin{matrix}\mathcal{D}_1 \\ A \\ \mathcal{D} \\ B\end{matrix}$$

  of B is valid.

- A closed proof of A not in an introduction form is valid if it reduces, by means of the given justifications, to a valid closed proof of A in an introduction form.

If we are only interested in closed proofs, this definition is sufficient. In view of the last clause, it is a generalized inductive definition proceeding on the complexity of end formulas and the reduction sequences generated by justifications. If we also want to consider open proofs, we would have to define:

- An open proof $\begin{matrix}A_1, \ldots, A_n \\ \mathcal{D} \\ B\end{matrix}$ is valid if for all closed valid proofs $\begin{matrix}\mathcal{D}_1 \\ A_1\end{matrix}, \ldots, \begin{matrix}\mathcal{D}_n \\ A_n\end{matrix}$, the proof $\begin{matrix}\mathcal{D}_1 \quad\quad \mathcal{D}_n \\ A_1, \ldots, A_n \\ \mathcal{D} \\ B\end{matrix}$ is a valid closed proof.

Given this clause for open proofs, the defining clause for the validity of a closed proof of $A \to B$ in introduction form might be replaced with

- A closed proof of $A \to B$ in the introduction form

$$\dfrac{\begin{matrix} \overset{(n)}{A} \\ \mathcal{D} \\ B \end{matrix}}{A \to B}\,(n)$$

  is valid if its immediate open subproof

$$\begin{matrix} A \\ \mathcal{D} \\ B \end{matrix}$$

  is valid,

yielding a uniform clause for all closed proofs in introduction form. However, this way of proceeding makes the definitions of validity of open and closed proofs intertwined, which obscures the fact that there is an independent definition of validity for closed proofs.

According to this definition, closed proofs are conceptually prior to open proofs. Furthermore, assumptions in open proofs are considered to be placeholders for closed proofs, as the validity of open proofs is defined by the validity of their closed instances obtained by substituting a free assumption with a closed proof of it. So we have identified two central features of standard proof-theoretic semantics:

The primacy of closed over open proofs (α)

The placeholder view of assumptions (β)

The definition of validity shows a further feature which is connected to (α) and (β). The fact that in an open proof $\begin{matrix} A \\ \mathcal{D} \\ B \end{matrix}$ the open assumption A is a placeholder for closed proofs $\begin{matrix} \mathcal{D}_1 \\ A \end{matrix}$ of A, yielding a closed proof

$$\begin{matrix} \mathcal{D}_1 \\ A \\ \mathcal{D} \\ B \end{matrix}$$

means that the validity of $\begin{matrix} A \\ \mathcal{D} \\ B \end{matrix}$ is expressed as the transmission of validity from [the closed proof] $\mathcal{D}_1$ to [the closed proof]

$$\begin{matrix} \mathcal{D}_1 \\ A \\ \mathcal{D} \\ B \end{matrix}$$

If one considers an open proof $\begin{smallmatrix} A \\ \mathcal{D} \\ B \end{smallmatrix}$ to be a proof of the consequence statement that B holds under the hypothesis A, this expresses

The transmission view of consequence (γ)

i.e., the idea that the validity of a consequence statement is based on the transmission of the validity of closed proofs from the premisses to the conclusion. This idea is closely related to the classical approach according to which hypothetical consequence is defined as the transmission of categorical truth (in a model) from the premisses to the conclusion. In that respect, Dummett–Prawitz-style proof-theoretic semantics does not depart from the classical view present in truth-condition semantics (see (Schroeder-Heister, 2008b)). Of course, there are fundamental differences between the classical and constructive approaches, which must not be blurred by this similarity, in particular with respect to epistemological issues (see (Prawitz, 2009)).[7]

A further point showing up in the definition of validity is the assumption of global reduction procedures for proofs (called "justifications"). This is what makes the (generalized) induction on the reduction sequence for proofs possible. It is assumed that it is not individual valid proof steps that generate a valid proof, but the overall proof which may *reduce* to a proof of a particular form (viz., a proof in introduction form). We call this

The global view of proofs (δ)

These four features are intimately connected to the model of natural deduction as its formal background. This holds especially for (β) and (δ), which specify (α) and (γ), respectively. Natural deduction permits to place a derivation on top of another one, and it is natural deduction where we have the notion of proof reduction. In the sequent calculus, this sort of connection is not present.

In the sequent calculus, logical inferences not only concern the right side of a sequent (corresponding to the end formula in natural deduction) but the right and left sides of sequents likewise. In this sense the sequent calculus is

[7] It might be mentioned that the definition of validity for a closed proof of $A \to B$ is closely related to Lorenzen's admissibility interpretation of implication. According to (Lorenzen, 1955), $A \to B$ expresses the admissibility of the rule $\frac{A}{B}$. The claim that every closed proof of A can be transformed into a closed proof of B can be viewed upon as expressing admissibility. At first glance, this contradicts the fact that in natural deduction an open proof $\begin{smallmatrix} A \\ \vdots \\ B \end{smallmatrix}$ is a proof of B *from* A and should as such be distinguished from an admissibility statement. However, even if, in the formal system, we are dealing with proofs *from* assumptions rather than admissibility statements, the semantic interpretation in terms of validity comes very close to the admissibility view. See (Schroeder-Heister, 2008a).

inherently *bidirectional* compared to the *unidirectional* formalism of natural deduction that underlies Dummett–Prawitz-style proof-theoretic semantics. In the following we will make a case for the bidirectional framework.

3 The sequent calculus and bidirectional natural deduction

According to the traditional, i.e. pre-natural-deduction reasoning model, we start with true sentences and proceed by inferences which lead from true sentences to true sentences. This guarantees that we always stay in the realm of truth.[8] Alternatively, we could start with assumptions and assert sentences under hypotheses. This is the background of natural deduction. Natural deduction adds the feature of discharging assumptions, i.e., the dependency on assumptions may disappear in the course of an argument. In this way the dynamics of reasoning not only affects assertions but at the same time the hypotheses assumed. However, this dynamics is very limited as the only options are introducing and discharging, so there is no more than a yes/no attribution to hypotheses. We cannot introduce and eliminate assumptions according to their specific meaning, which would be a more sophisticated dynamics. In this sense reasoning in standard natural deduction is assertion centred and unidirectional. This is even more so, as the hypotheses assumed are placeholders for closed proofs.[9]

A genuinely different model is given by the sequent calculus. The particular feature of this system, i.e. introduction rules on the left side of the sequent sign, can be philosophically understood as the meaning-specific introduction of assumptions. Consider conjunction with left sequent rules

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C} \qquad\qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

These rules can be interpreted as follows: Suppose we have asserted C under the hypotheses Γ and A. Then we may claim C by assuming $A \wedge B$ as an assumption and discharging A as an assumption, and similarly for B. Written in natural-deduction style this corresponds to the general elimination rules for conjunction

$$\frac{\overline{A \wedge B} \quad \begin{array}{c} (n) \\ A \\ \vdots \\ C \end{array}}{\mathrm{C}}\,(n) \qquad\qquad \frac{\overline{A \wedge B} \quad \begin{array}{c} (n) \\ B \\ \vdots \\ C \end{array}}{\mathrm{C}}\,(n)$$

[8] This was, for example, the picture drawn by Bolzano and Frege.

[9] This is not essentially changed if we replace assertion with denial and in this sense dualize natural deduction. Unidirectionality would just point into the opposite direction. See (Tranchini, 2009).

but with the crucial modification that the major premiss must now be an assumption, i.e., must occur in top position[10] (this is here indicated by the line over the major premiss). Similarly, the left implication rule

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}$$

is interpreted as follows: Suppose we have asserted both A under the hypotheses Γ, and C under the hypotheses Γ and B. Then we may claim C under the assumption $A \to B$ instead of B, i.e., discharge B and assert $A \to B$ instead. Written in natural-deduction style, this yields the general $\to$-elimination rule

$$\frac{\overline{A \to B} \qquad A \qquad \begin{matrix} (n) \\ B \\ \vdots \\ C \end{matrix}}{\mathrm{C}} \,(n)$$

again with the crucial difference to the standard general elimination rule that the major premiss occurs in top position.[11]

By presenting the sequent-calculus rules in a natural-deduction framework we are no longer working in "standard" or "genuine" natural deduction but in the reasoning model suggested by the sequent calculus, as the restriction on major premisses of elimination rules runs counter to the way premisses are treated in standard natural deduction. We call this modified system *bidirectional natural deduction* as it acts on both the assertion and the assumption side, with rules that depend on the forms of the formulas assumed or asserted. So the possible operations on assumptions are no longer merely structural.[12]

In proposing bidirectional natural deduction, as a natural-deduction-style variant of the sequent calculus, as our model of reasoning, we establish a symmetry between assertions and assumptions. Like assertions, assumptions can be introduced according to their meaning, namely as major premisses of elimination inferences. By imposing the restriction that major premisses must always be assumptions, elimination inferences receive an

[10] In Tennant's (Tennant, 1992) terminology, the major premiss "stands proud".

[11] A translation between sequent calculus and natural deduction with general elimination rules is carried out in full detail in (von Plato, 2001). Note that for implication, we are here considering the general elimination rule used by von Plato, as they correspond to the left sequent calculus rule, rather than the more powerful one proposed in (Schroeder-Heister, 1984), which extends the standard framework of natural deduction with rules as assumptions.

[12] We also call it "natural-deduction-style sequent calculus", as it is conceptually a sequent calculus which is presented in the form of a natural deduction system (Schroeder-Heister, 2004). In (Negri & von Plato, 2001), this term is used in a different sense, meaning a specific form of the sequent calculus.

entirely different reading. They are no longer justified by reference to the way the major premiss can be (canonically) derived. They are rather viewed as ways of introducing complex assumptions, given the derivations of the minor premisses. Elimination inferences in bidirectional natural deduction combine the introduction of an assumption with an elimination step and can thus be viewed as a special form of assumption introduction. Therefore we also call them "upward introductions", as opposed to "downward introductions" which are the common introduction rules.

Assumptions which are major premisses of elimination inferences are no longer placeholders for closed proofs as they cannot be inferred by means of an inference. They are always starting points of elimination inferences. Of course, it might be possible to show that given a proof $\begin{matrix}\mathcal{D}_1\\ A\end{matrix}$ of A and a proof $\begin{matrix}A\\ \mathcal{D}_2\\ B\end{matrix}$ of B from A, we can obtain a proof $\begin{matrix}\mathcal{D}\\ B\end{matrix}$ of B. However, this would have to be established as a theorem corresponding to cut elimination for the sequent calculus. It is no longer a trivial matter as in standard (unidirectional) natural deduction, since

$$\begin{matrix}\mathcal{D}_1\\ A\\ \mathcal{D}_2\\ B\end{matrix}$$

is no longer a well-formed proof if A is a major premiss of an elimination inference. Therefore bidirectionality overcomes the placeholder view of assumptions (β). With this it also overcomes the primacy of closed over open proofs (α) as closed proofs are no longer used to interpret assumptions. Only a premiss of an introduction rule can be viewed as a placeholder for a closed proof, which means that the uniform interpretation of assumptions by reference to closed proofs is given up.

The transmission view of consequence (γ) disappears as well. As assumptions can be introduced in the course of a proof (in the sequent calculus by left introduction, in bidirectional natural deduction as the major premiss of an elimination inference), it is no longer a defining feature of them that they transform closed proofs into closed proofs. If this happens to be the case, then it is "accidental" and has to be proved. The introduction of an assumption is just as primitive as the introduction of an assertion. In the terminology of Dummett–Prawitz-style proof-theoretic semantics, both the introduction of an assertion and the introduction of an assumption is a canonical, i.e. definitional step. More precisely, the distinction between canonical and non-canonical steps disappears. In this sense the concept of validity is much more rule-oriented than proof-oriented: We now consider a proof to be valid if it consists of proper applications of right and left rules (in the sequent calculus) or downward and upward introduction rules (in bidirectional natural deduction) rather than if it reduces to a proof in introduction form for all its closed instances. In this way, the global view of proofs (δ) also disappears, as it is based on the fundamental assumption that proofs are primary to rules and that the validity of rules is based on proofs and proof reduction. The idea of bidirectional reasoning is very much local rather than global.[13]

This does not mean that right and left (sequent calculus), or downward and upward (bidirectional natural deduction) introductions are unrelated to each other. We will still require some notion of harmony between the two sorts of inferences as an adequacy condition. However, this harmony will be local rather than global, and not based on proof reduction. One criterion would be *uniqueness* in the sense of (Belnap, 1961/62), which means that if we duplicate rules for a constant $*$, yielding a constant $*'$ with the same right (or downward) and left (or upward) rules, we can prove $A[*] \dashv\vdash A[*']$ in the combined system. There $A[*]$ is any expression containing $*$, and $A[*']$ is obtained from $A[*]$ by replacing $*$ with $*'$. However, unlike Belnap, we would not rely on conservativeness, as this is a global concept, but rather on local inversion in the sense that the defining conditions for a constant $*$ can be obtained back from this constant. Our main criticism of Belnap's proposal of conservativeness and uniqueness in his discussion of the connective "tonk" is that he mixes a local condition (uniqueness) with a global one (conservativeness).[14]

4 Why going local?

Why should we switch to a concept of hypothetical reasoning which is different from the standard one characterized by (α)–(δ), and which is prevailing both in classical and constructive semantics? The lack of an intuitive justification of the principles (α)–(δ) is no reason for abandoning them, if we cannot also tell why the bidirectional alternative has greater explanatory power. In fact, we gain access to a much wider range of phenomena, if we stick to the bidirectional paradigm. We just mention two points.

Atomic reasoning and inductive definitions

The discussion in proof-theoretic semantics has traditionally focused on logical constants. Logical constants are a particularly well-behaved case where we can apply the global considerations characteristic of the standard approach. Natural-deduction-based proof-theoretic semantics has been devel-

[13] The local approach to hypothetical reasoning put forward here was originally proposed by Hallnäs (Hallnäs, 1991, 2006).

[14] This point will be worked out elsewhere.

oped as a semantics of logical constants. However, this focus is much too narrow. Proof-theoretic definitions of logical constants just feature as particular cases of inductive definitions. Looking at inductive definitions as basic structural entities that confer meaning to objects, the distinction between atomic and non-atomic (i.e. logically compound) objects disappears. Most generally, we would deal with definitional clauses of the form

$$a \Leftarrow C$$

where a is an object to be defined and C is a defining condition. Starting with a definition of this kind, right (downward) and left (upward) introduction rules can be generated from this inductive definition in a canonical way, representing a way of putting inductive definitions into action, and resulting in powerful closure and reflection principles. The form of definitional clauses look like clauses in logic programming, and logic programs can be viewed as particular cases of inductive definitions. We would even generalize the framework set up by logic programming by considering clauses where the body C of a clause may contain hypothetical statements and therefore negative occurrences of defined objects. This goes beyond standard definite Horn clause programming and even transcends the classical field of logic programming with negation (Hallnäs & Schroeder-Heister, 1990/91). It differs from systems investigated in (Martin-Löf, 1971) in that it is not mainly directed at induction principles but rather the local inversion of rules. Systems of this kind have recently been considered by Brotherston and Simpson (Brotherston & Simpson, 2007), where also the relationship between inversion-based reasoning and induction principles for iterated inductive definitions is discussed. Considering inductive definitions in general opens up a wider perspective at hypothetical reasoning which is no longer based on logical constants. It can also integrate subatomic reasoning in the sense of (Więckowski, 2008), where the validity of atomic sentences is reduced to certain assumptions concerning predicates and terms.

Non-wellfounded phenomena

The global reductionist perspective underlying unidirectional natural deduction excludes non-wellfounded cases such as the paradoxes. The inductive definition of validity expects that there is no loop or infinite descent in the reasoning chain. However, in the case of the paradoxes, we have exactly this situation. Our local framework can easily accommodate such phenomena. For example, if we define p by $\neg q$ and q in turn by p, then both p and q are locally defined. The global loop is irrelevant for the local definition. In such a situation we can no longer prove global properties of proofs such as cut elimination, but this we do not require.

As it is now a matter of (mathematical) fact rather than a definitional requirement whether certain global properties hold, we do not rule out non-wellfounded phenomena by definition. This is a great advantage, as it gives us a better chance to understand them. Following (Hallnäs, 1991), we might call this approach a *partial* approach to meaning. According to Hallnäs, this would be in close analogy to recursive function theory, where it is a potential mathematical result that a given partial recursive function is total, rather than something that has to be established for the function definition to make sense.

There are other applications of the local approach that we cannot mention here such as the proper treatment of substructural issues, generalized inversion principles, evaluation strategies in extended logic programs, etc.

5 Final Digression: Dialogues

We have pleaded for a bidirectional view of reasoning as it is incorporated in Gentzen's sequent calculus and can be given the form of bidirectional natural deduction. As there are certain adequacy conditions governing such a system that relate right/downwards and left/upwards rules with one another, so that they are linked together in a certain way, we might ask of whether it would be possible to obtain them from a single principle. One possible answer might be the dialogical approach proposed by Lorenzen (Lorenzen, 1960) and his followers. If one carries its ideas over to the case of inductive clauses

$$\begin{array}{c} a \Leftarrow C_1 \\ \vdots \\ a \Leftarrow C_n \end{array}$$

one would be lead to an approach where an attack on the defined object a would have to be defended by a choice among the defining conditions C_i, which are themselves attacked by choosing one of its components. The distinction between right and left rules would then be obtained by strategy considerations for and against certain atoms. In this way a more unified approach could be achieved. The dialogical motivation, as based on local attack and defence rules, would not involve global reductive features compared to validity notions in standard proof-theoretic semantics. Therefore, it appears to be more faithful to our local approach, as the global perspective is only introduced at a later stage in terms of strategies and their transformations. In this way the dialogical research programme promises a novel perspective at the local/global distinction.

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