

Popper's Theory of Deductive Inference and the Concept of a Logical Constant

PETER SCHROEDER-HEISTER

Fachgruppe Philosophie, Universität Konstanz, Postfach 5560, 7750 Konstanz, W. Germany

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This paper deals with Popper's little-known work on deductive logic, published between 1947 and 1949. According to his theory of deductive inference, the meaning of logical signs is determined by certain rules derived from 'inferential definitions' of those signs. Although strong arguments have been presented against Popper's claims (e.g. by Curry, Kleene, Lejewski and McKinsey), his theory can be reconstructed when it is viewed primarily as an attempt to demarcate logical from non-logical constants rather than as a semantic foundation for logic. A criterion of logicity is obtained which is based on conjunction, implication and universal quantification as fundamental logical operations.

1. Introduction

Between 1947 and 1949 Popper published a series of articles on the foundations of (deductive) logic. The programmatic titles of some of them ('New foundations for logic', 'The trivialization of mathematical logic') indicate, as Popper said later in *1974b*, that they were written 'with much enthusiasm'. The immediate reaction of mathematical logicians to these papers was rather negative: reviews by Curry *1948a–1948d* and *1949*, Kleene *1948* and *1949*, and McKinsey *1948* pointed out several mistakes and expressed doubts as to whether Popper's programme could be retained.¹ There was, beside short (and positive) remarks by Kneale *1948* (see also his *1956*, 256 and *1961*, 102, and Kneale and Kneale *1962*, 563), no immediate reaction from more philosophically oriented logicians. The first article I know of is that of Merrill *1962*, which cites Prior's *1960* reference to Popper and Pap's *1958*, 144, footnote. Popper's theory of deductive logic is at least mentioned in Passmore *1957*, 407–408. There are further comments by Wisdom *1964* and Lenk *1970*, footnote 57.

The only detailed discussion published up to now of Popper's theory is Lejewski's *1974*. After that date I have only found Shaw's and Lyons's short statements *1977* on a special aspect of Popper's theory, Shoesmith's and Smiley's *1978*, 35, remark, Bartley's short discussion (*1980*, 69–70 and *1982*, 176–178), and some references in footnotes, in spite of the fact that the quantity of literature dealing with Popper's philosophy has been growing steadily from year to year. There exists some unpublished material, of which I know only Brooke-Wavell's Ph.D. thesis *1958* which

¹ See e.g. Curry *1949*: 'The whole program is very obscure, and has not been without serious error'. Beth's *1948* and Ackermann's *1948* and *1949a–1949b* reviews contain no criticisms but only statements of Popper's aims. Concerning Hasenjaeger's *1949* review see footnote 12.

presents a truth-functional discussion of Popper's logical rules without tackling the problem of their justification, and Dunn's Honors thesis 1963 discussing Popper's theory in some detail, partly following McKinsey 1948 in his criticisms. Judging by some remarks in Popper's reply 1974b to Lejewski's 1974 article in the Schilpp volume on *The philosophy of Karl Popper* (Schilpp 1974), it seems that the negative reactions or the lack of any reaction at all to his proposals kept Popper from further work in this direction.²

The aim of the present paper is to show that in spite of the correct criticisms by the above mentioned reviewers, Popper's articles contain some very interesting philosophical thoughts on the nature of deductive logic. More precisely, I think that they contain a detailed and well-founded proposal for an answer to the question of the distinction between logical constants and descriptive constants, or in Popper's terminology the distinction between formative and descriptive signs.

This problem has recently been discussed anew (cf. Bjurlöf 1978, Došen 1980, Field 1977, Hacking 1977 and 1979, Kuhn 1981, Lenk 1981, McCarthy 1981, Peacocke 1976, 1980 and 1981, Sundholm 1981 and Tennant 1980),³ partly in connection with growing interest in theories of meaning according to which meaning is determined by certain rules for the use of the signs considered. It may be that the main theme of Popper's articles fits in better with the present stage of discussion than with that in progress at the time of their appearance.

When we try to elaborate Popper's theory of what a logical constant is and to present it as a proposal worthy of discussion, it seems that we have, at least in part, to argue against his present view. In 1947b he states that the question of the demarcation of the logical constants was the main point he wanted to clarify (which is treated in Lejewski 1974 only marginally), but that he now thinks that it is insoluble and more particularly, that no sharp demarcation between logical and descriptive signs can be drawn. What we are going to show is that according to Popper's reconstructed theory the range of logical signs is indefinite and in this sense not sharply determined; however, the criterion we shall propose will make it possible to count certain signs as logical, and certain others as not logical, so that the demarcation line is not completely blurred.

In order to reconstruct Popper's theory of deductive logic as a theory which presents a definition of the logicity of signs we have to separate from it other aspects which have given rise to the rigorous and entirely correct criticisms in the above mentioned reviews. By this we mean in particular what may be called the foundational aspect, namely, Popper's programme to build up a basic semantics for logical signs by means of so-called 'inferential definitions'. Thus we think that the attempt to distinguish logical signs by presenting at the same time a semantical foundation for logic—an attempt made not only by Popper but also in most of the recent work on the problem of logicity (with the exception of Došen 1980)—fails,

2 Popper notes in 1974a, section 27, that Tarski was not interested in his work on formal logic. This was one of Popper's main reasons for giving up his work on logic (personal communication, 1982) although some unpublished material exists (cf. 1974a, note 198).

3 Further references can be found in Došen 1980. For the discussion of this problem in the German philosophical tradition see Lenk 1968.

but only because Popper's semantical foundation fails. We find that the arguments of the reviewers are directed only against the programme of a foundation of the laws of logic by 'inferential definitions'. It will turn out that once we give up this foundational claim of Popper's programme we can extract from his work a precise criterion of the logicality of signs.

In short, we shall interpret Popper's 'inferential definitions', which are intended to give a meaning to the logical constants, as *conditions* for the deducibility relations concerning certain signs. These must be fulfilled if these signs are to be regarded as logical, whereby the semantical justification of this deducibility concept itself must be given elsewhere and is independent of these conditions. This means of course that the idea of reducing logical laws to pure definitions and thus of trivializing logic—an idea which is often formulated in Popper's papers—can no longer be upheld. What remains is the more restricted but not less interesting problem of a definition of logicality independent of a semantical foundation for logic.

In the next section we treat Popper's theory as he himself presented it, while in the third section we deal with the main objections to this theory. In the fourth section we show how they can be avoided by abandoning Popper's foundational claims, and in the fifth section we discuss the definition of logicality which can be extracted from Popper's work after this reinterpretation of his programme. For simplicity we consider only sentential connectives as examples of logical operators. In the appendix we sketch the application of the proposed definition of logicality to quantifier logic (a case that is considered by Popper himself in *1947a–1947c*). The systematic investigation of specific constants for logicality according to the proposed definition as well as the comparison of logical constants in different logical systems (e.g. modal logic, relevant logic, quantum logic, higher order logic) must be left for further work; it would have to include an extensive discussion of Došen *1980*, which is the most highly developed systematic approach to the problem of logicality. Here we are mainly interested in establishing a systematic reconstruction of Popper's theory.

2. Popper's concept of 'inferential definitions' of the logical signs

Popper's starting-point is the assumption 'that the central topic of logic is the theory of formal or deductive inference' (*1947b*, 193). Thus it is one of the main tasks of logical theory to define the concept of a valid deductive inference (or 'consequence', which we use in the same sense). Now we have already such a definition at our disposal, viz. Tarski's model-theoretical one, whose roots go back to Bolzano. According to Tarski we can define an inference

$$\frac{A_1 \dots A_n}{A} \tag{2.1}$$

(where A_1, \dots, A_n, A are sentences) to be valid, if each model of A_1, \dots, A_n is also a model of A .

A *model* of a sentence A is defined as a sequence of objects which fulfils a sentential function A' , where A' is obtained from A by replacing all non-logical

constants of A by variables which correspond to them according to a fixed bijection between non-logical constants and variables (cf. Tarski 1936). So the definition of a valid inference presupposes the distinction between logical and non-logical constants.

The slightly modified way of defining the validity of an inference which Popper proposes is: (2.1) is valid if every form-preserving interpretation of (2.1) whose premisses are true has a true conclusion. A form-preserving interpretation is a mapping between languages ‘which (a) preserves the meaning of all the formative signs, i.e., gives a proper translation of all the formative signs, and which (b) preserves recurrences of those groups of non-formative (descriptive) expressions which, in a proper translation, would fill the spaces between the translated formative signs’ (1947a, 258). Thus this version too, which uses the concept of truth in an unanalysed way instead of the concept of fulfilment of a sentential function, takes for granted the distinction between logical and non-logical constants.

So the situation for Popper is that once we can draw a distinction between logical and descriptive signs a satisfactory definition of logical inference is possible along the lines of Tarski’s theory (cf. 1947a, 273).⁴ We could therefore expect Popper to give a definition of the logicity of a sign in order to supplement Tarski’s definition of a valid inference. Popper, however, goes quite another way, which he describes as follows (1974b, 1096):

Tarski showed that the concept of logical consequence can be easily elucidated [...] once we have decided upon a list of logical or formative signs. My idea was very simple: I suggested we take the concept ‘logical consequence’ as primitive and try to show that those signs are logical or formative which can be defined with the help of this primitive concept.

This formulation suggests that the definition of logicity presupposes a concept of logical consequence, which itself must not be dependent on such a definition (if we want to avoid circularities of definition). And this seems to mean that the definition of logical consequence has to come first, and then the definition of logicity, or that the latter has to use a sign for logical consequence as a kind of variable. The question of logicity would then no longer be important for the definition of logical consequence, which is taken by Popper to be one of the main tasks of logical theory, as we have seen.

This, however, is a somewhat misleading view. The concept of deducibility is Popper’s central concept and the logical operators obtain their meaning from roles they play in deducibility relations. In this sense Popper’s programme is completely different from the Tarskian approach, according to which the deducibility relation is

4 This had already been stated by Tarski 1936, to whom Popper refers (1947a, 251; 1974b, 1096). The concept of truth itself is not taken to be problematic by Popper (cf. 1947a, 265–267); in several writings beyond the articles considered here he emphasizes that he conceives of Tarski’s explication as successful. Nevertheless, the theory of deductive inference developed by Popper in order to avoid the problematic distinction between logical and non-logical signs in the definition of logical inference does not need the concept of truth, since it is based on rules.

justified with regard to logical operators with a previously given meaning. It is much more related to Gentzen's approach, whose natural deduction or sequent calculi are intended to characterize the meaning of the operators involved.⁵ This, however, does not mean that the deducibility concept is completely undefined or taken to be given elsewhere. The rules which determine the meaning of a sign (and whose special properties make the sign a logical one) also characterize the concept of deducibility. There is no non-trivial deducibility concept that does not employ rules for certain operators. So it would be more adequate to say that the logicity of signs is explained in a *framework* of a rule-governed semantic theory which is likewise explained, rather than to say that a deducibility concept is presupposed as undefined (or as defined elsewhere).⁶

When presenting Popper's theory we use '*a*', '*b*', '*c*', '*d*' and '*e*', with and without subscripts, as syntactical variables ranging over sentences of an object language which remains unspecified and which may differ from context to context. We consider only sentences since we have restricted ourselves to sentential logic. It is assumed that for the object languages considered a deducibility relation is given. We also speak of 'derivability' which is used in the same sense; a similar situation holds for 'deducible'/'derivable'. Deducibility can be considered to be a 2-place relation between finite and non-empty sets $\{a_1, \dots, a_n\}$ of sentences and sentences *a*. Following Popper we use the stroke '/' as the basic metalogical relation sign to express this relation:

$\{a_1, \dots, a_n/a\}$ means that *a* is deducible from $\{a_1, \dots, a_n\}$.

When we say that *a* is deducible from *a*₁ or from *a*₁, . . . , *a*_{*n*-1} and *a*_{*n*} we mean that *a* is deducible from $\{a_1\}$ or $\{a_1, \dots, a_n\}$. When mentioning metalogical signs and formulas we omit quotation marks, since it will be always obvious from the context whether they are used or mentioned. Without indication we undertake some slight technical improvements of Popper's formulations (e.g. by taking / to express a relation between a *set* of sentences and a sentence instead of a relation between an indefinite number of sentences and a sentence).

$a//b$ means the interdeducibility of *a* and *b*, i.e. a/b and b/a . Obviously // is an equivalence relation. Inference rules can be represented as

5 Cf. Gentzen 1935, 189 (ed. Szabo, pp. 80–81). The close relation of his framework to Gentzen's seems not to have been seen by Popper; only one short remark on Gentzen can be found in 1948a, 181.

6 Some passages show that in spite of treating the consequence-relation in quite another framework Popper does not entirely give up Tarski's concept of logical consequence. In 1947a, 287, he says that it is a problem, whether his new definition of the validity of an inference actually guarantees the transmission of truth. This can be understood as if Popper required a soundness proof of his concept of a logical consequence with respect to that of Tarski. This would coincide with statements often to be found in Popper's other writings that transmission of truth (or equivalently retransmission of falsity) is a central characteristic of logical inferences (cf. e.g. 1974a, sections 27 and 32). But he does not point out exactly the relationship between his alternative concept of derivability and Tarski's. Also in 1947b, 203 (footnote), Popper states that his theory 'opens a way to applying Tarski's concept without difficulty'. This seems also to be the result of Merrill's defence 1962 of Popper's theory against McKinsey.

$$(a_{11}, \dots, a_{1n_1}/a_1 \& \dots \& a_{m1}, \dots, a_{mn_m}/a_m) \rightarrow b_1, \dots, b_k/a, \quad (2.2)$$

where $\&$ and \rightarrow are the metalogical conjunction and implication signs, respectively. m may equal 0, i.e. the antecedens including \rightarrow may be missing. Thus Popper does not write rules as instructions regarding how one can produce signs from signs already produced, but describes metalinguistically the effect such instructions have on the deducibility relation. (Note that $/$ is not, as Gentzen's sequent arrow is, a sign belonging to the object language.) The free variables in metalinguistic statements like (2.2) are of course understood as universally bound. So we say that a metalinguistic statement, in particular a rule of the form (2.2), *holds* or *is valid*, if the metalinguistic statement holds which is obtained from it by universally quantifying over all variables (i.e., in the case of (2.2), over all a 's and b 's).

As basic inference rules which describe the derivability concept independently of the use of logical signs, Popper uses what he calls the generalized principles of reflexivity and transitivity:

$$a_1, \dots, a_n/a_i \quad (1 \leq i \leq n) \quad (2.3)$$

$$(a_1, \dots, a_n/b_1 \& \dots \& a_1, \dots, a_n/b_m \& b_1, \dots, b_m/a) \rightarrow a_1, \dots, a_n/a \quad (2.4)$$

These are, with minor differences (cf. Lejewski 1974), the rules which according to Tarski define a consequence relation. Popper calls these rules (and rules derived from them) 'absolutely valid'. More exactly, he defines a semantic concept of absolute validity with respect to which these rules are complete.⁷

This deducibility concept, which is based on (2.3) and (2.4), is extended by principles governing the use of certain object-language operators which are intended to be justified as logical constants. The aim of these principles is to define logical constants in terms of the derivability operator $/$. In order to explain what kinds of principles Popper distinguishes we shall take the conjunction sign \wedge of the object language as an example. (Concerning our way of designating object-language signs, we adopt the following convention: Atomic signs like signs for operators, parentheses and commas, are used autonomously, i.e. designating themselves. A succession of metalinguistic signs denoting object-language signs or being syntactical variables for object-language signs denotes for all values of the variables the result of the concatenation of the signs denoted.)

For conjunction Popper first proposes the principle: 'The statement c is a conjunction of the two statements a and b if, and only if, c/a , c/b and $a, b/c$ ' (1947b, 206). This is obviously an explicit definition of a 3-place metalogical predicate ' \dots is a conjunction of \dots and \dots '. It may be abbreviated as:

7 We omit a presentation of Popper's definition of absolute validity, which is a special version of Tarski's validity concept, in this special case not depending on the concept of a logical sign. Popper also sketches a way to avoid employment of the truth-predicate (1947a, 274–281). He presents too another approach to absolutely valid rules which uses the conjunction sign. He claims that the second approach is deductively stronger than the first, a claim which was shown to be false by Curry 1948a and 1948c.

$$\text{Conj}(c,a,b) \leftrightarrow (c/a \ \& \ c/b \ \& \ a,b/c), \quad (2.5)$$

where \leftrightarrow is the metalinguistic biconditional. (Other metalinguistic constants besides $\&$, \rightarrow and \leftrightarrow , for which we use abbreviations, are negation \sim and universal quantification \forall .) (2.5), being simply an explicit definition, is independent of whether the language considered possesses a statement c being a conjunction of a and b for every two statements a,b in the sense of (2.5), and of how many such conjunctions of a and b exist. Whether a language contains a conjunction is determined as follows: 'A language L contains the operation of conjunction if, and only if, it contains, with every pair of statements, a and b , a third statement c which is the conjunction of a and b ' (1947b, 206). These different definitions show that Popper wants to distinguish between abstract definitions of logical signs and their application to particular languages.

That Popper here speaks of *the* conjunction presupposes that a conjunction of a and b is uniquely determined. But this cannot mean that if c_1 and c_2 are conjunctions of a and b , then c_1 and c_2 are identical sentences. 'Uniquely' means 'uniquely up to interdeducibility'; i.e. if c_1 and c_2 are conjunctions of a and b , then $c_1//c_2$. More formally,

$$\forall a,b,c_1,c_2 ((\text{Conj}(c_1,a,b) \ \& \ \text{Conj}(c_2,a,b)) \rightarrow c_1//c_2).$$

This can be proved by use of (2.5).

Thus, if the language considered contains the operation of conjunction, instead of regarding conjunctions of a and b we can use abstract terms with respect to interdeducibility $//$. These abstract terms may be denoted by $a \wedge b$. If we conceive the abstract terms as denoting equivalence classes we may say that $a \wedge b$ denotes the equivalence class with respect to $//$ of all conjunctions of a and b .⁸

When we want to use such abstract terms like $a \wedge b$ as sentences, we have to define how to apply our deducibility relations to them. Assuming that a 'substitutivity principle for logical equivalence' holds (cf. Popper 1947b, 203), i.e. that derivability is not affected if a sentence is replaced by a $//$ -equivalent of it, we can identify $a \wedge b$ with its arbitrary representatives, i.e. define:

$$a_1, \dots, a_n/a \wedge b \text{ iff for an arbitrary conjunction } c \text{ of } a \text{ and } b: a_1, \dots, a_n/c,$$

$$a_1, \dots, a \wedge b, \dots, a_n/d \text{ iff for an arbitrary conjunction } c \text{ of } a \text{ and } b: a_1, \dots, c, \dots, a_n/d.$$

The procedure just sketched means that we have enriched a language which contains the operation of a conjunction by a standard name $a \wedge b$ for the conjunction of each

8 A concept of abstraction with respect to an equivalence relation, which does not depend on the formation of equivalence classes, was introduced by Lorenzen 1955, section 10, and 1962. It has the advantage that it can also explain the formation of classes (and in particular of equivalence classes) as a result of abstraction.

pair a, b . Since the use of $a \wedge b$ in deducibility contexts is now explained, we can treat signs of the form $a \wedge b$ like usual sentences.

According to our introduction of $a \wedge b$, $c // a \wedge b$ expresses that c is interdeducible with a conjunction of a and b and therefore itself such a conjunction. So $c // a \wedge b$ means the same as $Conj(c, a, b)$, and we can write instead of (2.5):

$$c // a \wedge b \leftrightarrow (c/a \ \& \ c/b \ \& \ a, b \not\#). \quad (2.6)$$

(2.6) is considered by Popper to be a reformulation of (2.5), expressing more clearly ‘that what we have defined is not so much the conjunction of a and b but the precise logical force (or the logical import) of any statement c that is equal in force to a conjunction of a and b ’ (1947b, 208).⁹ (2.6) is called an ‘*inferential definition*’ of conjunction. Inferential definitions of other signs have a similar form; those for (object-language) implication $>$, classical negation $-$ and disjunction \vee , for example (I use Popper’s somewhat unusual symbols) are (1947b, 218):

$$a // b > c \leftrightarrow \forall a_1 (a_1/a \leftrightarrow a_1, b/c) \quad (2.7)$$

$$a // -b \leftrightarrow \forall a_1, b_1 (a, a_1/b \rightarrow (a, a_1/b_1 \ \& \ a_1, b_1/b)) \quad (2.8)$$

$$\downarrow \vee \quad a // b \not\# c \leftrightarrow \forall a_1 (a/a_1 \leftrightarrow (b/a_1 \ \& \ c/a_1)). \quad (2.9)$$

For some operators Popper gives different (but equivalent) inferential definitions; another inferential definition for \wedge is:

$$a // b \wedge c \leftrightarrow \forall a_1 (a/a_1 \leftrightarrow b, c/a_1).^{10} \quad (2.10)$$

The general schema which underlies the various inferential definitions Popper presents for sentential operators can be stated for an n -ary sentential operator S as follows:

$$a // S(a_1, \dots, a_n) \leftrightarrow \mathfrak{A}(a, a_1, \dots, a_n). \quad (2.11)$$

9 The treatment just presented (particularly of $a \wedge b$ as an abstract term) is mainly an interpretation of mine. I see no other way to understand Popper’s claim that (2.6) is identical with (2.5). This interpretation is suggested by Popper’s calling $a \wedge b$ a ‘variable name’ of the conjunction of a and b (1947b, 207) which I can interpret only as ‘abstract term’ in the sense explained above (cf. especially 1947b, 214, lines 22 to 27). (2.6) presupposes of course the existence of an operation of conjunction whereas (2.5) does not: this will be one of the points of criticism to be described in section 3.

10 Cf. 1947b, 218–219. In 1949 Popper emphasizes that his inferential definitions often use certain maximality or minimality features of compound expressions with respect to deducibility. According to (2.7), $b > c$ is, for example, the weakest sentence from which, with b , c is deducible. According to (2.10), $b \wedge c$ is the strongest sentence deducible from b, c . A systematic account of logical operators as forming sentences which are in certain aspects strongest and weakest sentences has been undertaken by Tennant 1978, 74–77 in his ‘principle of harmony’. For a discussion of this principle cf. Over 1979.

Here $\mathfrak{A}(a, a_1, \dots, a_n)$ is a formula of the metalanguage, containing no more free syntactical variables for sentences than a, a_1, \dots, a_n and $/$ as the only predicate sign (i.e. besides $/$ only logical operators of the metalanguage). One could, as Popper does, consider also inferential definitions containing on the right hand side operators of the object language for which inferential definitions are already available. This, however, is not essential as will be shown in section 5 at (5.6)–(5.11).

Analogously, we can formulate as a general schema for explicit definitions like (2.5):

$$\mathfrak{I}_S(a, a_1, \dots, a_n) \leftrightarrow \mathfrak{A}(a, a_1, \dots, a_n), \quad (2.12)$$

with the same restriction for $\mathfrak{A}(a, a_1, \dots, a_n)$. \mathfrak{I}_S is an $(n + 1)$ – place metalinguistic predicate, $\mathfrak{I}_S(a, a_1, \dots, a_n)$ may be read as ‘ a is an S -connection of a_1, \dots, a_n ’.

Popper distinguishes ‘characterizing rules’, i.e. inference rules characteristic for an operator from inferential definitions of the form (2.11). For \wedge these are the rules

$$a \wedge b/a \quad a \wedge b/b \quad a, b/a \wedge b \quad (2.13)$$

or alternatively

$$a \wedge b/c \leftrightarrow a, b/c \quad (2.14)$$

(which is a pair of rules), for implication the rules

$$a/b > c \leftrightarrow a, b/c, \quad (2.15)$$

and for classical negation

$$\neg a, b/\neg c \leftrightarrow c, b/a. \quad (2.16)$$

These characterizing rules, also called ‘primitive rules of derivation’, are rules which are considered to determine the meaning of the operators involved (1947b, 213–215); Popper even speaks of ‘contextual definitions’ (1947b, 217). Characterizing rules are rules of the kind we know from Gentzen-style systems, i.e. rules concerning in each case one characteristic operator.

So we have two kinds of metalinguistic expressions: definitions of form (2.12) and inferential definitions of form (2.11) on the one hand, and characterizing rules (which have no standard form) on the other hand. Popper mainly discusses inferential definitions which he considers to be equivalent to definitions of form (2.12). In many cases (for example the standard sentential operators) the inferential definition of an operator holds if and only if certain rules hold; then the rules are characterizing rules of the same operator for which the inferential definition is given. He also speaks of fully characterizing rules (1948b, 113). For example, the rules (2.13) hold iff (2.10) holds, so they are characterizing rules of the conjunction operator, inferentially defined in (2.10). In such a case the characterizing rules and the inferential definition are interchangeable (1947b, 219).

This does not mean that the relation between characterizing rules and an inferential definition is not significant: If certain rules for an operator are reducible to inferential definitions, these rules can really be considered to be definitions determining the meaning of this operator (1947b, 219). That is, these rules are not simply laid down (as Popper thinks is the case in customary systems of logic) but are consequences of inferential definitions of the signs involved, and vice versa. Precisely due to their relation to inferential definitions, fully characterizing rules are not arbitrary but possess a defining character. By showing that rules for an operator are fully characterizing rules relative to an inferential definition of the operator, Popper tries to give a semantical justification of these rules. And since this is possible for the usual operators of sentential and quantifier logic, he believes himself to have given a justification of logic. This is done by reducing rules to inferential definitions; so he can say that ‘we obtain the whole formal structure of logic from metalinguistic inferential definitions alone’ (1947c, 562). Thus, according to Popper, all rules of logic are derived from pure definitions of the logical operators, and logic becomes a trivial matter.

Logic can therefore be described as containing, besides absolutely valid rules, characterizing rules for operators which can be shown to be equivalent to inferential definitions. This suggests a criterion for the logicity of an operator: an operator, characterized by certain rules, is logical iff these rules are equivalent to an inferential definition of the operator. This is in fact the criterion Popper employs in 1948b in order to show that certain operators are not logical (1948b, 118; cf. section 5 below).¹¹

3. Objections to Popper’s theory

We now discuss some objections stated by reviewers of Popper’s articles and by Lejewski—objections that are wholly correct if one takes the theory as it stands. We classify the objections in three groups. As the standard example of a logical operator we have again chosen the conjunction sign.

3.1. *First objection (Kleene and Lejewski)*. Popper’s so-called ‘inferential definitions’, of which (2.6) is an example, are at the first glance not *explicit* definitions, in so far as a complex statement stands on the left hand side of the biconditional; it contains the sign //, which is not the sign to be defined. Popper is aware of this (though he sometimes speaks of ‘explicit definitions’; for example, 1947b, 218), when he says that not the conjunction itself is defined, but its logical force (1947a, 286; 1947c, 564). We could retain the claim of (2.6) being an explicit definition if we interpreted the left hand side $c // a \wedge b$ to be a singular statement saying that c is the conjunction of a and b . In that case we would interpret $c // a \wedge b$ as a three-place predicate $././. \wedge$ applied to c , a and b . $c // a \wedge b$ would then be synonymous with $Conj(c, a, b)$, and $a \wedge b$ would not be a separate sentence of the object language.

This interpretation, however, cannot be maintained. Popper uses, as Kleene points

11 For other aspects of Popper’s work which we are not discussing here, in particular for the relation of Popper’s approach to Tarski’s theory of consequence and a formalization and equivalence proof of both theories, see Lejewski 1974.

out in 1948, $a \wedge b$ as a substitute for c when he derives the characterizing rules (2.13) or equivalent rules from (2.6) (cf. Popper 1947c, 565). In this way he presupposes that a conjunction operator is already at our disposal. Thus Kleene shows that the inferential definition (2.6) differs from an explicit definition (2.5) in that (2.6) makes an existential assumption; thus (2.5) and (2.6) are not different formulations of the same statement, as Popper thinks. This corresponds to our above reconstruction of his claim that (2.6) be identical with (2.5) where $a \wedge b$ was interpreted as an abstract term for a conjunction of a and b (presupposing that conjunctions of sentences always exist). Thus if we want to use the inferential definition (2.6) in the way Popper does, we first have to show that a conjunction of two sentences always exists in the language considered. Popper, however, wants to justify certain rules as meaning-determining rules for conjunction by their relation to an inferential definition, and so he must not presuppose the existence of this operator in the application of the inferential definition. In this way, Kleene's argument can be viewed as directed against Popper's semantical intentions: The inferential definitions cannot provide a semantics for logical operators since they presuppose the existence of the logical operators with the required semantical properties in the object language.¹²

Lejewski's argument (1974, 644–645) is closely related to Kleene's. He discusses first Popper's example (cp. 1947a, 284) of an inferential definition of a one-place operator 'the opponent of', for which we use the abbreviation $opp(\cdot)$:

$$a // opp(b) \leftrightarrow \forall c(b/a \ \& \ a/c). \quad (3.1.1)$$

From (3.1.1) we can derive, substituting $opp(b)$ for a and using reflexivity and transitivity of $/$, $\forall c(b/c)$; hence any deducibility assertion could be obtained in the language considered. Thus the acceptance of (3.1.1) would make the object language inconsistent.¹³ Lejewski sharpens this example by defining 'the opponent*' of ($opp^*(\cdot)$) by

$$a // opp^*(b) \leftrightarrow \forall c(b/a \ \& \ \sim(b/c)). \quad (3.1.2)$$

The acceptance of (3.1.2) would even make our *metalanguage* inconsistent.

The reason for this result, according to Lejewski, is that it is wrong to propound (3.1.2) 'as a legitimate definition without first satisfying ourselves that, in accordance with our explicit or tacit presuppositions, for every statement b of L the opponent* of b was also a statement of L '. Thus the inferential definitions do not define a new operation but give at most a name to the result of an operation (like conjunction), which must first be proved to be existent in the language considered. So Lejewski again stresses the existential presupposition of operations in the application of

12 Hasenjaeger 1949 stresses the same point: 'Die Schlußfiguren werden als Gebrauchsdefinitionen für die logischen Verknüpfungen aufgefaßt und gewinnen dadurch den Charakter von metalogischen Existenzaxiomen, was Verf. [Popper] anscheinend übersieht'.

13 One should note the similarities between Popper's discussion of the operator 'the opponent of' and the discussions of Prior's operator 'tonk'. Cf. Prior 1960 and 1964, Stevenson 1960, Belnap 1962 and Wagner 1981.

inferential definitions. The characterizing rules derived from inferential definitions are no longer semantical rules defining an operator but describe properties of conjunctions etc. which already exist in a particular language.

The interpretation Popper gives of (3.1.1) suggests a hypothetical interpretation of inferential definitions. That (3.1.1) yields a contradiction in the object language is for Popper no reason to reject (3.1.1) as a correct inferential definition. He conceives the result as a proof that ‘no *consistent* language will have a sign for “opponent of *b*”’ (1947a, 284). And analogously, the metalinguistic contradiction resulting from (3.1.2) could be counted as a proof that *no language at all* will have a sign for ‘opponent of *b*’. Then inferential definitions are to be read as: ‘If an opponent of *b* always exists in *L* then (3.1.1)’, ‘If a conjunction of *b* and *c* always exists in *L*, then (2.6)’, etc. To accept this is to accept Lejewski’s view that the inferential definitions and characterizing rules do not determine the meaning of a sign but describe properties of operations (viz. opponents, conjunctions, . . .) in given languages and only give them a name. This is not unreasonable—it is in fact an original idea to consider compound sentences $S(a_1, \dots, a_n)$ as abstract terms represented by interdeducible sentences *b* which stand in a certain metalinguistically describable relation \mathfrak{I}_S to the arguments a_1, \dots, a_n —but it is in strong conflict with Popper’s claim that the inferential definitions and characterizing rules alone provide a semantics for logical signs.

3.2. *Second objection (McKinsey)*. Both the definiens of an explicit definition like (2.5) and an inferential definition like (2.6) contain the deducibility sign \vdash . Thus the logical sign to be defined is defined in terms of the deducibility relation \vdash . Hence \vdash must be definable independently of the logical sign to be defined. That, however, is not possible. Only absolute deducibility (i.e. deducibility on the basis of the rules of reflexivity (2.3) and transitivity (2.4)) is independent of inference rules for logical operators. This concept however, is too weak to give the intended meaning to the logical operators, as McKinsey 1948 shows by the example of Popper’s inferential definition (2.9) of the disjunction sign \vee . From (2.9) we obtain

$$a \vdash b \vee c \leftrightarrow (b \vdash a \ \& \ c \vdash a). \quad (3.2.1)$$

Now it can be shown that if \vdash is understood as absolute deducibility, $a_1, \dots, a_n \vdash d$ implies that *d* is identical with an a_i ($1 \leq i \leq n$). Thus, if we understand on the right hand side of (2.9) and (3.2.1) \vdash in the absolute sense, it follows that if *a* is interdeducible with the disjunction of *b* and *c*, *a* is identical with *b* as well as with *c*.

This completely counterintuitive conclusion shows that in his inferential definitions Popper must understand \vdash in the specific sense based not only on absolutely valid rules but also on inference rules for the logical operators.¹⁴ On the other hand, he wants to justify their rules by inferential definitions. Clearly this is circular. This reinforces the point of the first objection. Popper’s inferential definitions can be used

14 There are, however, some passages suggesting that logical operators are defined in terms of absolutely valid rules; see e.g. 1947a, 282.

to check whether a given concept of derivability for a given object language contains sentences which may be viewed as the result of applying operations of conjunction, disjunction etc. to other sentences, and they can introduce signs like $a \wedge b$ for such sentences, but they cannot be used to justify the derivability concept itself. For that further arguments are required.

3.3. *Third objection (Kleene)*. A definition like (2.5) is obviously a definition of a 3-place metalinguistic predicate ' \dots is a conjunction of \dots and \dots ' and not a formal definition of a sign within the object language itself. Similarly, an inferential definition like (2.6) or (2.10) is a metalinguistic characterization of \wedge in terms of deducibility, but not a formal definition or characterization in the object language. What could best be considered to be a formal definition of \wedge would be the characterizing rules (2.13) for \wedge ; though they are metalinguistic expressions too, they have the form of rules. We have seen that Popper in fact considers these rules to be meaning-determining, in so far as they are derived from inferential definitions. That this is insufficient has been shown by the two objections above.

Kleene 1948 presents a further argument, drawing attention to a passage in Popper 1947c, 569, where Popper compares his system with that of *Principia mathematica*. Popper states that if in *Principia mathematica* a formula is provable which is not intuitionistically valid and contains no negation sign (like Peirce's formula $((a > b) > a) > a$) the derivation of this formula in his (Popper's) own system has to use the definition of classical negation. Now according to Kleene this situation could not obtain if we had only formal explicit definitions of the logical operators. 'It should make no difference whatsoever, in the case of a formula containing only " $>$ ", whether or not the definition of classical negation has been stated' (Kleene 1948, 174). This is, one could add, because definitions (and rules derived from them) should be *non-creative*: rules of classical negation must not allow us to obtain new derivability assertions not containing classical negation, if classical negation is a defined operator. Another good example would be the case of the contradictory operator *opp* with (3.1.1) as its inferential definition. Characterizing rules for *opp* (for which we may choose the two rules $a/opp(a)$, $opp(a)/b$) would make the language inconsistent and are thus trivially non-creative.

So Popper's theory involves more than only explicit definitions. A reconstruction of his theory would, according to Kleene, have to make explicit the way of obtaining deducibility statements for the object language within his metalinguistic system. This would, as Kleene supposes, require an inductive definition of 'obtainability' of metalinguistic formulas from inferential definitions which would presuppose the logical apparatus for which Popper is trying to give a foundation.

In 1963 Popper discusses an analogous subject with respect to the calculus of probability, and allows definitions to be creative (he actually compares creative definitions with 'real' definitions in the traditional sense). This is, however, only a different usage of the term 'definition' and no way out of the problem, since creative definitions are no longer purely semantical stipulations but contain existence claims. Popper is, in 1963, aware of this when he says that the creativity of definitions of operators forming from elements of a set S new elements 'can be eliminated by adding

to our axioms a requirement demanding the existence of an element which has exactly the properties of the element defined by the previously creative definition' (1963, 183). The fulfilment of such existence claims, however, is not simply a matter of stating an axiom: they may be, as the examples of *opp* and *opp** show, fulfilled only within inconsistent languages or no languages at all. This leads again to the first objection above: Popper's theory *presupposes* existence assertions and can be upheld only by relativizing it to them.

The objections show that Popper's programme to trivialize logic by reducing it to pure definitions fails (at least in the way Popper attempts to do it). It is not surprising that mathematical logicians like Curry, Kleene and McKinsey were not, when reviewing it, convinced by Popper's theory.

4. The reconstruction of Popper's theory

Going back to the philosophical motivation behind Popper's approach, we see that it was not primarily his intention to trivialize logic by reducing it to definitions alone (an aim which has been shown by his critics to fail). His starting-point was the problem that Tarski's definition of logical consequence presupposed a division of all signs into logical and extra-logical ones, and, therefore, a criterion of the logicity of signs. And what Popper primarily wanted to present was exactly such a criterion. Had he seen himself in a position to give it without using the term 'logical consequence' or 'deducibility' he probably would not have written a paper on the foundations of logic, because then the foundations of logic could have been given by Tarski's definition of logical consequence supplemented by Popper's definition of the logicity of signs. But since Popper's proposal for a distinction of logical signs is given within the framework of a theory of deducibility, the problems of the definition of the logicity of signs and of the definition of logical consequence appeared to be closely connected. This may be the reason why Popper treated as one task both the definition of logicity and the provision of a foundation for the concept of logical inference. The aim, however, to give a foundation for logic and logical inference was a secondary goal dependent on the primary goal of defining logicity. The quotation from 1974b, 1094, given above in section 2, confirms this view.

Since the objections described in section 3 are directed against this secondary goal we have two possible ways of saving a major part of Popper's theory. Firstly, we can give an exposition of his definition of a logical constant which does not refer to the concept of deducibility; secondly, we can give an exposition which does refer to the concept of deducibility but not in such a way that a solution of the problem of defining deducibility is logical^{ly} presupposed or equivalent to the definition of logicity. The former possibility does not really exist since it would result in a theory completely different from Popper's; thus we must confine ourselves to the latter.

Our proposal will be to give a definition of logicity in Popper's sense within the framework of deducibility. We will not, however, use this definition in order to *justify* certain inference rules for logical operators and hence a certain deducibility concept but shall consider this concept to be given elsewhere. We shall accept his idea that an operator characterized by certain rules is logical iff these rules are related in a certain way to an inferential definition of this operator. But we do not accept his

claim that this at the same time *justifies* the characterizing rules as rules semantically defining the operator in question. Semantical justification of the characterizing rules requires further arguments.

So we regard the semantical question of how deducibility rules can determine (or even define) the meaning of a constant as a problem different from the demarcation problem for logical constants. We give up the idea that the metalinguistic characterization of signs as logical signs gives a meaning to any sign, or that the fact that certain rules are meaning-giving rules for an operator depends on its characterization as a *logical* one. (Conversely, we do in fact hold that, if we are given meaning-determining rules for a sign, then we can tell from those rules whether the sign is a logical one. The definition of logicality itself, however, does not consist in stating such rules but only in stating the necessary conditions such rules (or the deducibility relation generated by them) have to fulfil, in order to be counted as rules for logical signs.)

This does not imply that a theory of this kind is not possible. Kneale 1956, or Hacking 1977 and 1979, for example, define the logicality of a sign by the specific way it obtains its meaning (namely by introduction and elimination rules of a certain form); something similar can be said for other modern positions. But that would yield an entirely new theory and not a reconstruction of Popper's theory (we would have to discuss e.g. questions of non-creativity of characterizing rules and several other problems).

So we arrive at the following view. The meaning of logical operators can be described by certain characterizing rules governing the operators. The logicality of the operators is established by the fact that these characterizing rules obey principles of a certain form (inferential definitions). That the characterizing rules are meaning-determining does not follow from this fact (as Popper erroneously assumed that it does). What we can extract from Popper's theory is a criterion of logicality and not a satisfactory meaning theory.

That does not mean that Popper's criterion of logicality *presupposes* such a meaning theory. The idea that operators are logical if their rules fulfil inferential definitions may be formulated as follows. If we want to build up a meaning theory for logical operators, the resulting rules which describe the meaning of those operators should be of such a form that they fulfil inferential definitions. That is, Popper's theory formulates restrictions for a semantic theory if it is to be a semantic theory for *logical* signs. In other words, if we want to justify a certain deducibility concept within a semantic theory and want the signs this deducibility concept deals with to be logical signs, our semantic theory has to fulfil certain restrictions. The restrictions are given by the admissible formulations of inferential definitions.

From this point of view we may interpret the inferential definition of an operator *S* as an *adequacy condition* for a deducibility concept and rules describing the meaning of *S*. What Popper calls an inferential definition of e.g. \wedge can be conceived as an adequacy condition for a semantical justification of certain rules which are intended to represent the meaning of \wedge . An operator is logical if an adequacy condition of a certain form can be stated for its rules.

This view of inferential definitions as adequacy conditions for a semantical

justification of rules has the advantage that the definition of logicality does not refer to a special kind of semantical justification of rules. Popper's definition of logicality is compatible with a semantical justification e.g. of the rules of \wedge in a truth-condition-semantics, in a game-theoretical framework, in a Gentzen-Prawitz-style framework where meaning is determined by introduction rules for operators etc. It is only required that the result of such a semantical justification is formulated as a deducibility concept for sentences containing the operator in question. (We need not even require that this relation be recursively enumerable; in that case we should not speak of *rules* which characterize the meaning of an operator.) As opposed to this, all other approaches, aside from Došen 1980, to the question of logicality quoted in section 1 above use definite semantical frameworks which are not easy to compare with one another. And the end of the discussion as to which semantical framework to choose for an investigation of logical operators is not in sight. What makes Popper's definition of logicality (in our reconstruction) so interesting is its independence from these difficult questions of the foundations of semantics.¹⁵

Before going on to a more detailed exposition of Popper's definition of logicality we may sketch the way in which our proposal avoids the three objections mentioned in section 3.

Firstly, since inferential definitions are no longer considered to be meaning determining but instead adequacy conditions there is no reason to require them to be explicit definitions. Furthermore, it need no longer be assumed that inferential definitions of the conjunction and disjunction sign etc. presuppose that the language considered already contains conjunctions and disjunctions etc. in the sense of definition (2.5) or generally (2.12). Adequacy conditions may be unsatisfiable (whereas semantical determinations must be satisfiable by the operator they want to define). The inferential definitions of *opp* and *opp** are examples of adequacy conditions which cannot be fulfilled respectively by any given consistent language or by any language at all.

Secondly, that the inferential definitions presuppose the sign \wedge presents no difficulty. As adequacy conditions for the justification of a deducibility concept they *have to* contain a sign for deducibility. A definitional circularity can only obtain for real definitions, not for adequacy conditions.

Thirdly, that Popper's theory lacks a semantics for the *formal* signs considered and presents only metalinguistic explicit and inferential definitions is an objection only as long as one wants to provide a semantics for logical signs.

5. What is a logical constant?

We have shown that we can avoid the criticisms of Popper's theory if we put aside his claim to give a semantical foundation for logic and interpret his 'inferential definitions' as adequacy conditions for systems of rules for *logical* operators (to be justified semantically elsewhere). Now we want to discuss the definition of a logical

15 Our interpretation of inferential definitions as adequacy conditions does not affect the status of explicit metalinguistic definitions like (2.5). They can as before be used to introduce by abstraction e.g. an operator \wedge , if for sentences *a, b* a conjunction *c* (where *c* need not have a specific form) always exists. But this introduction of an explicit operator does not eo ipso provide a semantics for it.

constant resulting from this proposal. We present it in a systematic way (with respect to sentential operators), in some points deviating from Popper's ideas.¹⁶

We define an n -place sentential operator S to be a logical operator if

$$a//S(a_1, \dots, a_n) \leftrightarrow \mathfrak{A}(a, a_1, \dots, a_n) \quad (5.1)$$

holds, where $\mathfrak{A}(a, a_1, \dots, a_n)$ is a formula of the meta-language containing no more free syntactical variables for sentences than a, a_1, \dots, a_n , no name of a sign of the object language, and $/$ as the only predicate (besides metalinguistic logical signs). We use Popper's term 'inferential definition' for (5.1), though we are aware that we do not consider it to be a definition. Contrary to Popper, we do not require that (5.1) be equivalent to characterizing rules for S . (5.1) is an adequacy condition for rules for an operator S and for a corresponding deducibility concept; so it formulates deducibility relations involving the operator S which have to hold if we want to consider S to be a logical operator.

We can give a related definition of the logicity of $(n + 1)$ -place metalinguistic predicates \mathfrak{I}_S , defined explicitly by

$$\mathfrak{I}_S(a, a_1, \dots, a_n) \leftrightarrow \mathfrak{A}(a, a_1, \dots, a_n) \quad (5.2)$$

where $\mathfrak{A}(a, a_1, \dots, a_n)$ is of the form stated above. We define \mathfrak{I}_S to be a logical relation, if for all a_1, \dots, a_n there is an a , unique up to interdeducibility, such that $\mathfrak{I}_S(a, a_1, \dots, a_n)$ holds (with regard to the object language considered).¹⁷

Whereas the first definition defines the logicity of a sign of the object-language, the second definition defines the logicity of a relation between sentences. It is, however, immediately obvious that if S is a logical operator in the first sense, the relation between a, a_1, \dots, a_{n-1} and a_n which holds if a is interdeducible with $S(a_1, \dots, a_n)$ is a logical relation in the second sense. Conversely, if we have a logical relation \mathfrak{I}_S in the second sense, we can introduce an n -place operator S of the object language such that $S(a_1, \dots, a_n)$ represents as an abstract term all the a 's for which $\mathfrak{I}_S(a, a_1, \dots, a_n)$ holds (cp. section 2 above); S is then logical in the first sense. Logical operators are just the operators introducible from logical relations by abstractions with respect to interdeducibility. Thus it makes no difference for our purposes which definition of logicity we consider; we shall choose the first one.

It is important to note that the definition of logicity includes the uniqueness of the operator concerned. If we have

16 This can only be a sketch of some ideas, not a fully worked out theory. Such a theory has recently been presented in great detail by Došen 1980 and applied to various logical systems including modal and relevance logics. Došen's theory shares with the view presented here the emphasis on the point that questions of logicity must be sharply distinguished from questions of semantics.

17 The requirement that the first argument of \mathfrak{I}_S be unique up to interdeducibility could be included in definition (5.2) by formulating it as

$$\mathfrak{I}_S(a, a_1, \dots, a_n) \leftrightarrow \forall b(b//a \leftrightarrow \mathfrak{A}(b, a_1, \dots, a_n)).$$

$$a // S_1(a_1, \dots, a_n) \leftrightarrow \mathfrak{A}(a, a_1, \dots, a_n)$$

and

$$a // S_2(a_1, \dots, a_n) \leftrightarrow \mathfrak{A}(a, a_1, \dots, a_n)$$

for different n -place operators S_1 and S_2 , we can infer

$$S_1(a_1, \dots, a_n) // S_2(a_1, \dots, a_n).$$

So operators which are not determined uniquely in this sense by their semantical rules cannot be counted as logical according to our definition. This will be the case with negation in Johansson's minimal logic (see below).¹⁸

In the following we have to show how far the proposed definition of logicity is an adequate explication of what we associate with such a concept. Furthermore, we must defend it against possible objections. We start with an objection concerning the deducibility concept.

The definition of logicity refers to a framework in which the meaning of operators is given by certain deducibility relations. That does not mean that the semantics of an operator must be given in terms of deducibility or of rules, but only that the semantics justifies in the end certain deducibility relations, whatever kind of semantics may be applied. Nonetheless, the definition of logicity does not leave completely open what properties the deducibility relation $/$ may have. That Popper takes the generalized rules of reflexivity (2.3) and transitivity (2.4) as basic might suggest that they must be supposed to be features of the framework. This again would rule out important arguments against the general validity of these rules—as developed e.g. in the discussion on relevant implication and the logic of entailment—from consideration. However, it is not necessary to regard the generalized rules of reflexivity and transitivity to be parts of the chosen framework.

What is required in essential arguments (e.g. for the equivalence of our two forms of definitions of logicity and the uniqueness of operators fulfilling the same inferential definition) is solely that

$$// \text{ is an equivalence relation,} \tag{5.3}$$

18 One can even prove the stronger result: that if

$$\begin{aligned} a // S_1(a_1, \dots, a_n) &\leftrightarrow \mathfrak{A}_1(a, a_1, \dots, a_n), \\ a // S_2(a_1, \dots, a_m) &\leftrightarrow \mathfrak{A}_2(a, a_1, \dots, a_m) \end{aligned}$$

and

$$\mathfrak{A}_1(a, a_1, \dots, a_n) \rightarrow \mathfrak{A}_2(a, a_1, \dots, a_m)$$

hold, then

$$S_1(a_1, \dots, a_n) // S_2(a_1, \dots, a_m).$$

This means, for example, that if classical negation \neg and intuitionistic negation \neg are both present in a system, we have $\neg a // \neg a$. Cf. Popper 1948b, 113–114.

and

$$/ \text{ is invariant with respect to } //, \quad (5.4)$$

i.e.

$$\text{if } a_1, \dots, a_i, \dots, a_n/a \text{ and } a_i//b, \text{ then } a_1, \dots, b, \dots, a_n/a,$$

and

$$\text{if } a_1, \dots, a_n/a \text{ and } a//b, \text{ then } a_1, \dots, a_n/b.$$

These are very weak properties, and it is hardly possible to dispense with them since they are involved in the concept of 'uniqueness up to // with respect to deducibility'. These conditions are fulfilled e.g. in the system of entailment of Anderson and Belnap 1975.

That no more conditions are required for the supposed deducibility relation / has consequences for the formulations of inferential definitions of operators, if we want stronger principles for / to hold. The classical and intuitionistic operators as they are used in the ordinary systems of natural deduction require a framework in which the generalized principles of reflexivity and transitivity hold, so these principles must be added to the inferential definitions. As an inferential definition of conjunction in classical or intuitionistic logic (2.10) is no longer sufficient; one should take e.g.

$$a//b \wedge c \leftrightarrow [Vc_1(a/c_1 \leftrightarrow b, c/c_1) \& \mathfrak{B}], \quad (5.5) \quad \mathcal{V}$$

where \mathfrak{B} contains (2.3) and (2.4).¹⁹

Besides the deducibility concept with some minimal properties the definition of logicality presumes of course the distinction between sentences and other linguistic entities, in so far as use is made of variables for sentences of the object language. This, however, is no real objection. A constant is always a constant of a certain syntactical type, and in order to define the logicality of a constant we may presuppose a certain distinction between these types. The logicality of a sign depends on certain (adequacy) conditions for its semantics, and the latter obviously must assume syntactical distinctions to be given. Also the fact that, when adding the principles of reflexivity and transitivity to inferential definitions we have to use variables for finite sets of sentences, presents no problem, since finite sets of sentences are not more problematic than sentences themselves.

The restriction that in (5.1) $\mathfrak{A}(a, a_1, \dots, a_n)$ must not contain any operator of the object language (more precisely: names for such operators) provokes a further objection. It seems to make sense to allow $\mathfrak{A}(a, a_1, \dots, a_n)$ to contain operators which have already been shown to be logical. In Popper 1947b, 219, we find e.g. an inferential definition of the 'alternative denial' λ of the form

19 Popper's own attempt to connect the principles of reflexivity and transitivity with the inferential definition of λ (cf. 1947c, 565) contains some errors, as Curry 1948 pointed out.

$$a//b\lambda c \leftrightarrow a//-(b \wedge c) \quad (5.6)$$

where $-$ and \wedge are operators for which inferential definitions are given by (2.8) and (2.10). However, (5.6) can be transformed into an equivalent inferential definition whose right hand side uses no operator. We can eliminate $-$ by first transforming (5.6) to

$$a//b\lambda c \leftrightarrow \forall d(\forall a_1, b_1(d, a_1/b \wedge c \rightarrow (d, a_1/b_1 \& a_1, b_1/b \wedge c)) \rightarrow a//d) \quad (5.7)$$

and then transforming this to

$$\begin{aligned} a//b\lambda c &\leftrightarrow \forall e(\forall c_1(e/c_1 \leftrightarrow b, c/c_1) \\ &\rightarrow \forall d(\forall a_1, b_1(d, a_1/e \rightarrow (d, a_1/b_1 \& a_1, b_1/e)) \rightarrow a//d)). \end{aligned} \quad (5.8)$$

Using the inferential definitions (2.8) and (2.10) and the principles for $//$ and $/$ mentioned above, (5.6), (5.7) and (5.8) can easily be proved to be mutually equivalent.

In general, let an (improper) inferential definition

$$a//S(a_1, \dots, a_n) \leftrightarrow \mathfrak{A}(a, a_1, \dots, a_n) \quad (5.9)$$

be given containing within $\mathfrak{A}(a, a_1, \dots, a_n)$ an occurrence of $S_1(A_1, \dots, A_m)$ as an argument of $/$, where A_1, \dots, A_m may be built up from sentential variables and possibly further operators. Let a (proper) inferential definition

$$a//S_1(a_1, \dots, a_m) \leftrightarrow \mathfrak{A}_1(a, a_1, \dots, a_m) \quad (5.10)$$

be given, where $\mathfrak{A}_1(a, a_1, \dots, a_m)$ fulfils our restrictions. Then (5.9) is obviously equivalent to

$$a//S(a_1, \dots, a_n) \leftrightarrow \forall b(\mathfrak{A}_1(b, A_1, \dots, A_m) \rightarrow \mathfrak{A}'(a, a_1, \dots, a_n)) \quad (5.11)$$

where $\mathfrak{A}'(a, a_1, \dots, a_n)$ is the result of substituting b for the considered occurrence of $S_1(A_1, \dots, A_m)$ in $\mathfrak{A}(a, a_1, \dots, a_n)$. This equivalence contains at least one occurrence of $S_1(A_1, \dots, A_m)$ less than (5.9). By repeated application of this procedure we can eliminate from $\mathfrak{A}(a, a_1, \dots, a_n)$ each operator different from S and arrive at a proper inferential definition in our sense. This reduction only uses the above mentioned principles postulated for $/$. This is, however, only a proof sketch. For the inductive proof one has to choose an induction value which takes into account also the complexity of $S_1(A_1, \dots, A_m)$.

The most serious objection to the proposed definition of a logical operator is that it takes for granted the distinction between logical and non-logical operators in the metalanguage when we require that the right hand side $\mathfrak{A}(a, a_1, \dots, a_n)$ of an inferential definition (5.1) may contain besides $/$ only *logical* operators. We could avoid this circularity if we did not speak of 'logical operators' in general, but of specific

operators, e.g. \forall , \rightarrow , $\&$ and \sim ; our requirement would then be that $\mathfrak{A}(a, a_1, \dots, a_n)$ may contain besides / only these specific operators. Here the term 'logical operator' no longer appears. There would be no circularity if we used certain operators (e.g. \rightarrow) in the metalanguage to state the conditions an operator of the object language (e.g. $>$) has to fulfil in order to be logical: we may *use* the metalinguistic sign \rightarrow to formulate an inferential definition for the sign $>$ of the object-level. In doing so we would not employ the distinction between logical and metalogical signs on which the conditions for inferential definitions rest.

This is what Popper seems to have in mind when he emphasizes that the *use* of logical expressions in the metalanguage does not conflict with the definitions of the corresponding signs in the object language, since they are signs of another language (1947a, 288–289; 1947b, 233–235; cf. Kneale 1948, 158). The argument is quite correct, but it holds only as long as we do not want to justify why we choose *this* metalogical definition of logicality referring to *these* specific operators and not another one allowing other signs. And it is this *justification* which requires the distinction between logical and non-logical signs.

So we have reached the point at which we must give positive reasons why the proposed definition of logicality explicates what we mean intuitively with this concept. Popper himself says nothing on this important point. We have to argue that the definition of logicality transmits the idea of *topic-neutrality* usually connected with the concept of logicality.²⁰ That means in the present context, where we take the meaning of a sign to be given by certain deducibility relations, that the validity of these relations does not depend on properties specific to some context; or at least that the operator considered is uniquely determined already (up to interdeducibility) by deducibility relations which can be described without reference to any 'material' context. This is exactly what an inferential definition does: It describes the deductive meaning of an operator by purely logical concepts and requires that this meaning is determined uniquely.²¹ That is, the definition of logicality preserves the idea of logicality because the metalinguistic operators which inferential definitions contain besides / are purely logical. So not the formulation but the justification of the definition of logicality has to use the distinction between descriptive and logical signs. Thus we arrive at a circle which is not a *circle of definition* but a *circle of justification*: even if we define the logicality of a sign of the object language not by using the term 'logical sign of the metalanguage' but by referring to certain signs of the metalanguage leaving open whether they are logical or not, in the *justification* of the definition we have to argue that they are logical. (This seems to be the point of the argument of Shaw and Lyons 1977.)

20 The term 'topic-neutral' seems to have been introduced by Ryle 1954, 115–116, as cited in Dunn 1963.

21 So the uniqueness condition implicit in inferential definitions is of great importance. If an inferential definition could be fulfilled without guaranteeing the uniqueness of the operator considered we could extend the meaning of that operator by empirical conditions without affecting the fulfilment of the inferential definition. The uniqueness condition means that no 'real' extension of the meaning of an operator, as given by deducibility relations which are described in logical terms, is possible, i.e. that the meaning of the operator is in some way maximalized. Negation in Johansson's minimal calculus, for example, lacks this uniqueness and can thus be shown to be non-logical in the sense of our definition (see below).

Now this circularity is not such a catastrophe as it might at first glance appear, since we do not always have to use the logical constant in question in the metalinguistic inferential definition for a certain logical constant. If we look at the inferential definitions for conjunction, implication, classical negation and disjunction (2.6) or (2.10), (2.7), (2.8) and (2.9), we see that they use only the positive logical constants \forall , $\&$ and \rightarrow , and this remains true even if we consider quantifiers (see section 6), and also for other types of negation such as the intuitionistic one. This shows that at least a *reduction* of the question of logicality to the set $\{\forall, \&, \rightarrow\}$ of constants is possible: once we have a justification of these three constants as logical ones we can justify many other constants as logical ones by means of Popper's criterion.

This reduction is not unimportant, in particular with respect to negation. It shows that negation, often considered to be a very 'problematic' constant, is only of secondary significance for this point; the main problem is that of justifying the 'less problematic' signs \forall , $\&$ and \rightarrow as logical signs.²²

The fact that in his consideration of constants of modal logic Popper uses metalinguistic disjunction, and that further signs perhaps require negation in the formulation of their inferential definitions (cf. 1947c, 570) does not affect the primacy of \forall , $\&$ or \rightarrow . Since we can establish the logicality of disjunction and negation with respect to \forall , $\&$ and \rightarrow , we may use them *in the next step* to justify certain other signs as logical and so on. The picture we obtain of the range of logical signs within Popper's framework can thus be described as follows: The basic logical signs are \forall , $\&$ and \rightarrow . They cannot be justified as logical signs within the framework itself. By means of \forall , $\&$ and \rightarrow the logicality of other operators, especially disjunction, negation in various versions, existential quantification and equality can be established. The additional use of these operators may justify further operators as logical, and so on. In this way we obtain a hierarchy of logical signs whose basis consists of \forall , $\&$ and \rightarrow and in which the 'lower' operators may be used to justify the 'higher' operators as logical. There is no reason to suppose that this hierarchy is of a definite height since at each stage we obtain new means of expression which may be incorporated in inferential definitions for logical signs.²³ This destroys a conception, held in particular by Kant and his followers, that a *complete* account of all logical constants is possible (see Lenk 1968).

In this way we obtain a plausible interpretation of Popper's theory not based on circular argumentation. It must of course be supplemented by a justification of \forall , $\&$ and \rightarrow as logical constants. Since we are here concerned with a reconstruction of

22 Popper does not emphasize very strongly the principal importance of this reduction to certain basic signs and only says that 'this fact may be interesting for some reason or other' (1947b, 235); he mentions, however, in 1947c, 563, how few logical means are needed in the metalanguage. Carnap 1942, 59–60, points out that the solution of the problem of demarcation for logical operators within general semantics might be solved if the problem is solved for the metalanguage, but he does not take into consideration the possibility of a reduction to certain central logical operators.

23 This does not contradict the result sketched above at (5.6)–(5.11) that improper inferential definitions containing signs already introduced can be turned into proper ones containing no operators; it concerned only operators of the object language but not the logical signs of the metalanguage which are considered here.

Popper's theory, we shall give only a short and rather tentative sketch of the direction in which such a justification might lie.

First, however, we present an example of an alleged logical sign which is in fact not a logical sign according to our definition. This shows that not each sign whose use can be described by rules as a logical sign in the proposed sense; it is not enough simply to encode the intuitions one has about a certain sign into the form of rules in order to get a logical sign. We choose the negation sign of Johansson's 1937 minimal calculus and reconstruct a proof given by Popper that it is not a logical sign (cf. Popper 1948b, 116–118). We have to modify Popper's proof somewhat since we do not assume the *equivalence* between characterizing rules and inferential definitions.

The negation sign considered, which we note as \Leftarrow , is characterized by the single rule

$$(a, b/c \ \& \ a, b/ \Leftarrow c) \rightarrow a/ \Leftarrow b \quad (5.12)$$

(constructive 'reductio ad absurdum').²⁴ We suppose furthermore the generalized rules of reflexivity and transitivity to be given. To show that \Leftarrow is not a logical sign we have to show that \Leftarrow does not fulfil

$$a// \Leftarrow b \leftrightarrow \mathfrak{A}(a, b) \quad (5.13)$$

for any $\mathfrak{A}(a, b)$ satisfying our conditions. Now consider the one-place operator $+$ characterized by the rule

$$a/ + b. \quad (5.14)$$

$+$ is a logical operator since it fulfils the inferential definition

$$a// + b \leftrightarrow \forall c(b/a \leftrightarrow c/a),$$

or shorter

$$a// + b \leftrightarrow \forall c(c/a).$$

(For the following argument it is not important that $+$ is logical.) From (5.14) we obtain the rule

$$(a, b/c \ \& \ a, b/ + c) \rightarrow a/ + b \quad (5.15)$$

(taking for granted the law $A \rightarrow (B \rightarrow A)$ in the metalanguage), which is the complete

²⁴ Popper states this rules wrongly as

$$a, b/ \Leftarrow c \rightarrow a, c/ \Leftarrow b,$$

but that does not affect his argument.

analogue of (5.12). If it were possible to prove (5.13) on the basis of (5.12), it would also be possible to prove

$$a// + b \leftrightarrow \mathfrak{A}(a,b) \quad (5.16)$$

on the basis of (5.15) and thus (5.14). (5.13) and (5.16) yield, in a system having (5.12), (5.14), generalized reflexivity and transitivity as rules,

$$\leftrightarrow b// + b. \quad (5.17)$$

In the same way we would obtain

$$\leftrightarrow b// - b \quad (5.18)$$

for classical negation — (the same would also hold for intuitionistic negation), since the analogue of (5.12) can be obtained from the rules of classical negation. (5.17) and (5.18) yield

$$- b// + b,$$

from which

$$a/b, \quad (5.19)$$

i.e. the inconsistency of the system, follows. However, consistent systems with both the operators — and + can easily be construed. (Already (5.17) would yield a contradiction, since it implies $a/\leftrightarrow b$.)

A closer inspection of this argumentation shows that it is the lack of uniqueness which makes \leftrightarrow a non-logical operator (cf. Curry 1948d). The single rule (5.12) holds both for classical negation — and the trivial operator +, and the uniqueness requirement which would be contained in an inferential definition of \leftrightarrow would make them equivalent. This would no longer be possible if we added to (5.12) e.g. an elimination rule for \leftrightarrow like

$$a, \leftrightarrow a/b \quad (\text{'ex contradictione quodlibet'}).$$

But then we would already have an intuitionistic negation.

A case comparable with \leftrightarrow would be an operator . having the single rule

$$a,b/a.b, \quad (5.20)$$

i.e. conjunction with the elimination rule omitted. This operator could be proved in an analogous way not to be logical, since both for conjunction \wedge and a two-place operator \circ , characterized by the rule

$$a/b \circ c$$

and fulfilling the inferential definition

$$a//boc \leftrightarrow \forall d(d/a),$$

rules

$$a,b/aob \text{ and } a,b/a \wedge b$$

analogous to (5.20) would hold. This would, if . were logical and had an inferential definition,

$$a//b.c \leftrightarrow \mathfrak{A}(a,b,c)$$

yield inferential definitions

$$a//boc \leftrightarrow \mathfrak{A}(a,b,c)$$

and

$$a//b \wedge c \leftrightarrow \mathfrak{A}(a,b,c)$$

which would imply

$$a \wedge b//aob.$$

From this again (5.19) would follow. Popper (1948b, 116–118) gives further examples of negations for which no inferential definitions exist.

Finally, I want to sketch roughly what a justification of \forall , $\&$ and \rightarrow as basic logical operators could look like. I am not yet clear about these matters, so the thoughts presented are very preliminary in character. First, it seems that we must define 'logicality' positively and not define 'descriptivity' first and then 'logical' as 'non-descriptive' (as Carnap 1942, 57–58, proposes for 'special semantics'). That is because the borderline of 'descriptive' is too hazy. Furthermore, it is doubtful whether 'descriptive' and 'logical' allow a complete division of all expressions: minimal negation \leftrightarrow for example, which was shown to be non-logical, can hardly be counted as a descriptive sign.

To solve our problem by simply stating certain rules for \forall , $\&$ and \rightarrow and defining them to be logical rules is by no means satisfactory; what we want to know is *whether* and *why* such rules are logical. From a criterion for logicality in the style of the one given above we cannot expect more success: It would have, too, to be a criterion which does not use the operators \forall , $\&$, \rightarrow or other operators, since we want to justify the logicality of other operators relative to the logicality of \forall , $\&$ and \rightarrow . Thus it does not seem to be promising to look for a 'theoretical' criterion, i.e. a criterion which describes the logicality of an operator by properties of the deducibility relation. As an alternative I want to propose a more pragmatic criterion in order

to explain the topic-neutrality of \forall , $\&$ and \rightarrow . Instead of trying to give such an explanation in terms of the deducibility relation \vdash we could ask what is involved in all descriptions of the actions which underlie the deducibility concept. And if we could show that just the three basic operators underlie the description of such actions, their topic-neutrality would follow from the fact that no special content of action is presupposed.

So we ask what means of expression must be at our disposal if we want to describe the actions whose results are described metalinguistically by the deducibility relation. This is a somewhat 'transcendental' question: we have chosen as our framework that the meaning of operators is represented by certain deducibility relations; now we ask for the means to describe the actions which underlie these deducibility relations. If such a description were not possible, it would not make sense to assume the framework itself, since it would not have real significance.

The actions underlying the deducibility relations can be characterized as *rule-governed* activities; in ideal form they are present in the production of signs according to the rules for a formal calculus. This is immediately obvious if deducibility relations are semantically justified via the justification of certain rules, i.e. if deducibility relations simply describe the result of applying rules. But it holds also for other semantical frameworks that deducibility relations state which inferences are allowed, even if the rules themselves are not part of the justification of the deducibility relations.

Now any description of such action uses a kind of 'if...then', since it must express that *if* a certain action has been performed, *then* one may perform a certain other action. Since a new action may depend on the performance of *more than one* previous action, we must also have a kind of 'and' at our disposal: If actions A_1 and $A_2 \dots$ and A_n have been performed, then one may perform another action. Furthermore which actions may be performed is usually described only in the sense of certain *types* of action and not as concrete actions. This makes it necessary to use \forall : *for all* a_1, \dots, a_n, a of types A_1, \dots, A_n, A respectively, if a_1 and $\dots a_n$ have been performed, we may perform a .

If this is correct, \forall , $\&$ and \rightarrow are topic-neutral, since operators of this kind are involved in each description of a deducibility concept \vdash , which was the basis of all our preceding discussions. In other words: If \forall , $\&$ and \rightarrow had a specific material content, they could not be used for the description of these most general actions presupposed by any deducibility concept. An elaboration of this approach has to state in detail which kinds of rules hold for the operators \forall , $\&$ and \rightarrow so justified. The result would be, as far as I see, the rules of positive logic for these operators, i.e. the usual introduction and elimination rules for them. (This is also mentioned by Popper in *1947b*, 235 and *1947c*, 563.)²⁵ There is no reason to take the stronger classical rules (which make e.g. Peirce's law derivable) and also no reason to choose a 'relevant' or related version of \rightarrow . It may be that 'relevant implication' or similar operators can be shown

25 Because of the very restricted metalinguistic use of \forall and \rightarrow , Popper draws an analogy to Hilbert's programme: 'Hier liegt in der Tat etwas Ähnliches vor wie in Hilbert's Programm, eine finite Begründung für einen nicht-finiten Kalkül zu finden' (personal communication, 1982).

to be logical at a later stage of the theory and it may be that there are strong semantic reasons for entailment logic. But the primary logical constants underlying each description of rule-governed activities seem to be the usual ones of positive logic.²⁶

6. Appendix: first-order quantifier logic

In order to extend the proposed definition of logicality to (first-order) quantifiers we first have to state what kinds of operators quantifiers are. One possibility is to say that they are variable-binding operators which are applied to formulas and yield a formula. The other possibility is to consider quantifiers to be operators which are applied to predicate terms and yield a formula. Here a predicate term (I take this expression from Prawitz 1965, 63) is an n -place expression which, when applied to individual terms t_1, \dots, t_n , yields a formula. A special case of a predicate term is an n -place predicate which yields a closed formula, i.e. a sentence, when applied to n individual terms. This way presupposes an operation of building predicate terms from formulas (e.g. by λ -abstraction); the quantifiers themselves, however, do not bind variables. According to the first possibility, we obtain the formula $\Lambda_x A(x,y,z)$ from the formula $A(x,y,z)$ by universal quantification with respect to x , where x is no longer free in the resulting formula. According to the second possibility, we obtain the formula $\Lambda(\lambda x.A(x,y,z))$ from the predicate term $\lambda x.A(x,y,z)$ by universal quantification. Here the operation of binding the variable x is contained in the formation of $\lambda x.A(x,y,z)$ from the formula $A(x,y,z)$ and not in the application of the universal quantifier. The use of λ -abstraction and the interpretation of λ -terms as predicate terms obviously presuppose a rule of λ -conversion like

$$(\lambda x.A)(t) = A[x/t] \text{ for all individual terms } t,$$

where $A[x/t]$ denotes the result of the substitution of t for x in A .

For the present purpose it seems to be better to choose the second approach. Then we can dispense with problems arising from relabelling bound variables, from the definition of an operation of substitution etc., because we need not deal with λ -abstraction explicitly in this context. We only need to have syntactical variables $a^i, b^i, a_i^j, b_i^j, \dots$ for i -place predicate terms and t, t_1, t_2, \dots for individual terms, and to know that $a^i(t_1, \dots, t_i)$ is a formula for an i -place predicate term a^i and individual terms t_1, \dots, t_i . As a special case we have a^0, b^0, \dots as variables for formulas. We do not need to determine explicitly how to form predicate terms from formulas or how the application of predicate terms to individual terms works, i.e. we need not refer

26 It is interesting that the negation sign plays no basic role. That shows that the central paradigm is that of deductive reasoning under the aspect of supporting statements by others and not under the aspect of refutation. According to Popper this means that logic is considered primarily to be the 'organon of deduction, or proof' rather than the 'organon of rational criticism' (Popper in Schilpp 1974, 1081), that is, it is considered with respect to 'its use in the demonstrative sciences, that is to say, the mathematical sciences' rather than to 'its use in the empirical sciences' (Popper 1970, 18 = 1972, 305). Although these aspects are closely connected, we may say that in dealing with formal logic Popper does not (directly) employ the apparatus of his philosophical views developed in particular in his philosophy of science. Lenk 1970 proposes some ideas as a basis for a foundation for logic within the framework of rational criticism. There, of course, negation becomes a basic sign.

explicitly to a rule of λ -conversion. In particular, it is not necessary to have an operation of substitution at our disposal. Such an operation would of course appear in a particular formal system of quantifier logic, but it need not appear in inferential definitions, i.e. in the conditions on which the logicity of a sign depends. What remains in the definition of the logicity of signs is the unspecified application of predicate terms to arguments.

Thus we avoid all the problems Popper 1947b has with his concept of substitution. He states rules for this concept which he takes to be implicit definitions of it (1947b, 194, footnote 1). He even uses formulations which suggest that he thinks substitution to be a *logical* operation for which an inferential definition can be stated (1947b, 225). An inspection of his rules shows, however, that they cannot be brought into the form of an inferential definition of an operator of the object language. If at all, Popper's rules can be considered to be an implicit characterization of a meta-linguistic operation. (In this way substitution is conceived in 1947c, 563.) Besides this there are some more other questionable points in his treatment of substitution and quantification; for instance, his notion of '*a* does not depend on *x*' ($a_{\bar{x}}$) (1947b, 226), and his inferential definitions of quantifiers which do not have the form

$$'a // \dots \leftrightarrow \dots', \text{ but } 'a_{\bar{x}} // \dots \leftrightarrow \dots' \text{ or } 'a(\bar{x}) // \dots \leftrightarrow \dots'$$

(1947b, 228–229 and Corrections 69–70). Popper himself saw the problems connected with his concept of substitution, but an improved version of his theory has never been published (cf. 1974a, section 27 and note 198).

Arguments of $/$ are now formulas and not only sentences. That is because we have to take into account a deducibility concept in which the assertion of quantified formulas depends on the schematic derivability (i.e. derivability with free variables) of certain formulas. This concerns sentential operators too, since they may occur in the scope of a quantifier. Particularly, individual terms may include individual variables.

With a logical operator S , we have to associate not only a number n of places but an n -tuple $\langle \alpha_1, \dots, \alpha_n \rangle$ of numbers which may be called its *type* and means that the i -th argument of S must be an α_i -place predicate term. So we have: for an operator S of type $\langle \alpha_1, \dots, \alpha_n \rangle$ and n predicate terms A_1, \dots, A_n whose numbers of argument places are $\alpha_1, \dots, \alpha_n$ respectively, $S(A_1, \dots, A_n)$ is a formula. An n -place sentential operator now becomes an operator of n -ary type $\langle 0, \dots, 0 \rangle$.²⁷

As a schema for an inferential definition of an operator S of type $\langle \alpha_1, \dots, \alpha_n \rangle$ on which the logicity of S depends we can formulate:

$$a^o // S(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}) \leftrightarrow \mathfrak{A}(a^o, a_1^{\alpha_1}, \dots, a_n^{\alpha_n}), \quad (6.1)$$

where $\mathfrak{A}(a^o, a_1^{\alpha_1}, \dots, a_n^{\alpha_n})$ is characterized as (5.1) with the sole addition that the variables $a_i^{\alpha_i}$ can stand for different kinds of entities. (6.1) includes, as a special case, schema (5.1) for n -place sentential operators if α_i equals 0 for all i ($1 \leq i \leq n$).

27 If we had chosen the above mentioned first possibility and treated quantifiers as operators binding variables within formulas, each α_i within a type $\langle \alpha_1, \dots, \alpha_n \rangle$ would have to be not simply a number but a sequence $x_{i_1}, \dots, x_{i_{\alpha_i}}$ of variables, which are exactly those variables that become bound in the formula being the i -th argument of S .

As examples of inferential definitions of quantifiers we may state for the universal quantifier Λ (of type $\langle 1 \rangle$):

$$a^0 // \Lambda(b^1) \leftrightarrow \forall c^0 (c^0 / a^0 \leftrightarrow \forall t (c^0 / b^1(t)))$$

and for the existential quantifier \forall (of type $\langle 1 \rangle$):

$$a^0 // \forall(b^1) \leftrightarrow \forall c^0 (\forall t (b^1(t) / c^0) \leftrightarrow a^0 / c^0).$$

More complicated examples are inferential definitions for the quantifier I of type $\langle 1, 1 \rangle$, forming particular affirmative judgments in the sense of traditional syllogistics:

$$a^0 // I(b^1, c^1) \leftrightarrow \forall e^1 (\forall t (e^1(t) / b^1(t) \& e^1(t) / c^1(t) \& b^1(t), c^1(t) / e^1(t)) \rightarrow \forall d^0 (\forall t (e^1(t) / d^0) \leftrightarrow a^0 / d^0))$$

or for the quantifier $\Lambda \forall$ of type $\langle 2 \rangle$ with the intended meaning $\Lambda_x \forall_y$:

$$a^0 // \Lambda \forall(b^2) \leftrightarrow \forall e^1 (\forall t_1 \forall c^0 (\forall t (b^2(t_1, t) / c^0) \leftrightarrow e^1(t_1) / c^0) \rightarrow \forall d^0 (d^0 / a^0 \leftrightarrow \forall t (d^0 / e^1(t))).$$

Finally, we sketch how this kind of analysis can be extended to identity. Identity is an operator which has two individual terms as arguments and has formulas as values. Its inferential definition can be stated as follows:

$$a^0 // t_1 = t_2 \leftrightarrow (\forall b^1 (a^0, b^1(t_1) / b^1(t_2)) \& \forall c^0 (\forall b^1 (c^0, b^1(t_1) / b^1(t_2) \& c^0, b^1(t_2) / b^1(t_1)) \rightarrow c^0 / a^0)). \quad (6.2)$$

Rules fulfilling (6.2) are e.g. those given in Hacking 1978, 618.²⁸

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28 Popper's inferential definitions of identity using the substitution operation (1947b, 228 (cf. Corrections, 69, and 1947c, footnote 1), 1948b, 112, footnote 11) correspond to (6.2). However, they are not quite correct, since they omit the second conjunct in the parentheses left to $\rightarrow c^0 / a^0$. It is, however, not necessary to have $\forall b^1 (a^0, b^1(t_2) / b^1(t_1))$ as an additional conjunct on the right hand side of (6.2).

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