Classical Lambek Logic

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Abstract. We discuss different options for two-sided sequent systems of non-commutative linear logic and prove a restricted form of cut elimination.

By "classical Lambek logic" we denote a sequent system with sequences of propositional formulas on the right and left side of the sequent sign, which has no structural rule except cut. We credit this logic to J. Lambek since he was the first to investigate Gentzen-systems without structural rules — originally in an intuitionistic setting, i.e. with not more than one formula in the succedent of a sequent, and motivated by linguistic considerations (see [4]). From the point of view of linear logic classical Lambek logic can be considered as pure (i.e., without exponentials) noncommutative (i.e., without the structural rules of exchange) classical (i.e., multiple succedent) linear propositional logic. This is the starting point of Abrusci's [1] paper. Abrusci presents a sequent calculus together with a semantics in terms of phase spaces. By proving completeness he gives a semantic justification of the sequent system. Independently, under the heading "bilinear logic" Lambek himself has studied this system (see [6]) based on categorical considerations.

The following investigation is purely proof-theoretic. In the first part we will attempt to give a proof-theoretic motivation of the sequent-systems Abrusci and Lambek propose. This motivation is not obvious since the cut rule, the negation rules and the rules for the binary multiplicatives have to be formulated with restrictions (side conditions) which at first sight do not seem natural at all. We will consider different possible forms of these rules and show that essentially two systems, $\mathcal{A}$ (corresponding to the calculi proposed by Abrusci and Lambek) and $\mathcal{B}$ are possible, which by permutation of antecedents or succedents of sequents can be embedded into each other. It can be shown that the unrestricted rules one would expect from usual sequent systems make exchange derivable. So our criterion for the acceptance of a rule is that its side conditions preclude the derivation of exchange, but are as weak as possible in that respect.

In the second part of this paper we deal with cut elimination. Abrusci has given an example which shows that even the restricted cut rule cannot be eliminated from the sequent system considered. We show that cut elimination holds if we confine ourselves to sequents in which in the scope of a negation sign no connective except the same negation occurs.

Part I: Sequent Systems for Classical Lambek Logic

We write sequents in the form $\Gamma \vdash \Delta$, where Γ and Δ stand for sequences of formulas. When no confusions can arise, we write these sequences without commas. Expressions like $\Gamma_1 \Delta_1$ or $\Delta_1 A B \Delta_2$ are then understood in the obvious way.

1. The invalidity of unrestricted cut

In his thesis, Gentzen [2] formulates the cut rule in the following way (the differences being just notational):

$$(\mathrm{Cut}_G)\ \frac{\Gamma \vdash \Delta_1 A \quad A \Gamma_2 \vdash \Delta}{\Gamma \Gamma_2 \vdash \Delta_1 \Delta}.$$

Here the cut formula A occurs in rightmost position in the left premiss and in leftmost position in the right premiss. This is no loss of generality since, due to the presence of exchange rules

$$\frac{\Gamma \vdash \Delta_1 A B \Delta_2}{\Gamma \vdash \Delta_1 B A \Delta_2} \qquad \frac{\Gamma_1 A B \Gamma_2 \vdash \Delta}{\Gamma_1 B A \Gamma_2 \vdash \Delta},$$

Gentzen can assume that an A occurring somewhere in the succedent of the left premiss can be permuted to the rightmost position, and an A in the antecedent of the right premiss can be permuted to the leftmost position. Since in classical Lambek logic we discard all structural rules, particularly exchange, this can no longer be assumed. In this logic, the natural unrestricted formulation of cut covering (Cut_G) as a special case would be the following:

$$(\mathrm{Cut}_U)\frac{\Gamma \vdash \Delta_1 A \Delta_2 \quad \Gamma_1 A \Gamma_2 \vdash \Delta}{\Gamma_1 \Gamma \Gamma_2 \vdash \Delta_1 \Delta \Delta_2}.$$

However, it turns out that this formulation is too strong. Firstly it violates the principle that two consecutive cuts in a derivation should be permutable[1]. Although the following derivations result from each other only by permuting cuts, their end sequents differ from each other with respect to the order of B_1 and B_2 — in the absence of exchange rules a crucial difference:

$$\frac{\dfrac{\Gamma \vdash A_2 A_1 \quad A_1 B_1 \vdash \Delta_1}{\Gamma B_1 \vdash A_2 \Delta_1} \qquad A_2 B_2 \vdash \Delta_2}{\Gamma B_1 B_2 \vdash \Delta_2 \Delta_1} \qquad \frac{\dfrac{\Gamma \vdash A_2 A_1 \quad A_2 B_2 \vdash \Delta_2}{\Gamma B_2 \vdash \Delta_2 A_1} \qquad A_1 B_1 \vdash \Delta_1}{\Gamma B_2 B_1 \vdash \Delta_2 \Delta_1}.$$

Secondly, (Cut_U) makes certain rules of exchange derivable, if negation is available. To show that, we need not discuss the precise form of negation rules in classical Lambek logic here but just assume that at least one of the following four pairs of negation axioms is available:

$$\begin{array}{llllll} (\mathrm{N1}) & \vdash A \neg A & A \neg A \vdash & (\mathrm{N3}) & \vdash \neg A A & A \neg A \vdash \\ (\mathrm{N2}) & \vdash A \neg A & \neg A A \vdash & (\mathrm{N4}) & \vdash \neg A A & \neg A A \vdash . \end{array}$$

If (N1) holds, then with unrestricted cut we obtain the following derivations:

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[1] We cannot discuss this principle here but mention only that it has proved quite useful, e.g. in proofs of cut elimination (see Girard's proof in [3, p. 109]) and as an equality law in a categorical setting of sequent systems (see Lambek [5]).

$$\dfrac{\dfrac{\vdash B\neg B \quad AB\Gamma\vdash\Delta}{A\Gamma\vdash\Delta\neg B} \qquad B\neg B\vdash}{BA\Gamma\vdash\Delta} \qquad\qquad \dfrac{\vdash B\neg B \qquad \dfrac{\Gamma\vdash AB\Delta \quad B\neg B\vdash}{\Gamma\neg B\vdash A\Delta}}{\Gamma\vdash BA\Delta} \quad ,$$

which yield forms of exchange in the antecedent and in the succedent. Similar derivations can be given for any other pair of negation axioms (N2), (N3) or (N4).

2. Possible formulations of the cut rule

Both Abrusci [1, p. 1414] and Lambek [6, p. 220] have proposed the following restricted cut rule:

$$(\mathrm{Cut})_{\mathcal{A}} \; \frac{\Gamma\vdash\Delta_1 A\Delta_2 \quad \Gamma_1 A\Gamma_2\vdash\Delta}{\Gamma_1\Gamma\Gamma_2\vdash\Delta_1\Delta\Delta_2} \qquad \text{provided } \Delta_2\Gamma_1 \text{ or } \Delta_1\Gamma_2 \text{ or } \Delta_1\Delta_2 \text{ or } \Gamma_1\Gamma_2 \text{ empty}.$$

This means that the cut rule splits up into the following four unrestricted rules:

$$(\mathrm{Cut.1})_{\mathcal{A}} \; \frac{\Gamma\vdash\Delta_1 A \quad A\Gamma_2\vdash\Delta}{\Gamma\Gamma_2\vdash\Delta_1\Delta} \qquad (\mathrm{Cut.3}) \; \frac{\Gamma\vdash A \quad \Gamma_1 A\Gamma_2\vdash\Delta}{\Gamma_1\Gamma\Gamma_2\vdash\Delta}$$

$$(\mathrm{Cut.2})_{\mathcal{A}} \; \frac{\Gamma\vdash A\Delta_2 \quad \Gamma_1 A\vdash\Delta}{\Gamma_1\Gamma\vdash\Delta\Delta_2} \qquad (\mathrm{Cut.4}) \; \frac{\Gamma\vdash\Delta_1 A\Delta_2 \quad A\vdash\Delta}{\Gamma\vdash\Delta_1\Delta\Delta_2} \; .$$

The first rule is Gentzen's rule (Cut_G), the second one a dualized version of it with the cut formula A standing in the opposite outermost positions. Rule three is just an adaptation of the unproblematic intuitionistic (single-succedent) cut rule to the classical case, and rule four is its dual with right and left sides interchanged. Since we would always expect the last two rules to hold, the crucial cases are the first two. In the following they will also be called the "critical" as opposed to the "uncritical" cases of cut. We call the cut rule with these four subcases "$(\mathrm{Cut})_{\mathcal{A}}$" to distinguish it from another possible cut rule "$(\mathrm{Cut})_{\mathcal{B}}$" to be discussed later on.

To justify $(\mathrm{Cut})_{\mathcal{A}}$, we show that the restrictions on cut are strong enough to prevent the deficiencies of unrestricted cut, and that weaker restrictions would again have undesired consequences.

Consecutive applications of $(Cut)_{\mathcal{A}}$ are permutable: Consider derivations of the following kind, where we use $\Gamma[A]$ or $\Gamma[A, B]$ to express that the formula A or the formulas A and B occur at certain places in the sequence of formulas Γ (A and B not necessarily next to each other):

$$\dfrac{\dfrac{\Gamma_1\vdash\Delta_1[A,B] \quad \Gamma_2[A]\vdash\Delta_2}{\Gamma_2[\Gamma_1]\vdash\Delta_1[\Delta_2,B]} \qquad \Gamma_3[B]\vdash\Delta_3}{\Gamma_3[\Gamma_2[\Gamma_1]]\vdash\Delta_1[\Delta_2,\Delta_3]} \qquad \dfrac{\dfrac{\Gamma_1\vdash\Delta_1[A,B] \quad \Gamma_3[B]\vdash\Delta_3}{\Gamma_3[\Gamma_1]\vdash\Delta_1[A,\Delta_3]} \qquad \Gamma_2[A]\vdash\Delta_2}{\Gamma_2[\Gamma_3[\Gamma_1]]\vdash\Delta_1[\Delta_2,\Delta_3]} \; .$$

If the restrictions of $(\mathrm{Cut})_{\mathcal{A}}$ are met in all four cuts in the two derivations, then, by checking cases, it is easy to see that $\Gamma_3[\Gamma_2[\Gamma_1]]$ is the same as $\Gamma_2[\Gamma_3[\Gamma_1]]$.

Only two pairs of negtion axioms are compatible with $(Cut)_{\mathcal{A}}$: If we combine $(\mathrm{Cut})_{\mathcal{A}}$ with (N1) or (N4), we obtain examples of exchange such as the following:

$$\dfrac{\dfrac{\vdash B\neg B \quad \Gamma AB\vdash}{\Gamma A\vdash \neg B} \qquad B\neg B\vdash}{B\Gamma A\vdash} \qquad\qquad \dfrac{\dfrac{\vdash \neg AA \quad AB\Gamma\vdash}{B\Gamma\vdash \neg A} \qquad \neg AA\vdash}{B\Gamma A\vdash}\ .$$

With (N2) or (N3) no such examples can be constructed. (This follows from Corollary 9 proved at the end of this paper.)

The provisos for (Cut)$_{\mathcal{A}}$ cannot be weakened: If we allow for less sequences of side formulas to be empty, we can again derive exchange. For instance, if we permit $\Delta_2\Gamma_1$ to be nonempty, then with any pair of negation axioms we can derive a form of exchange. As an example we consider (N3):

$$\dfrac{\dfrac{\vdash \neg BB \quad AB\Gamma_2\vdash \Delta}{A\Gamma_2\vdash \neg B\Delta} \qquad B\neg B\vdash}{BA\Gamma_2\vdash \Delta}$$

So the restrictions of (Cut)$_{\mathcal{A}}$ are optimal in a certain sense.

Another version of cut: There are other possible restrictions of cut which cannot be compared directly with (Cut)$_{\mathcal{A}}$. Such restrictions lead to (Cut)$_{\mathcal{B}}$:

$$(\mathrm{Cut})_{\mathcal{B}}\ \dfrac{\Gamma\vdash \Delta_1 A\Delta_2 \quad \Gamma_1 A\Gamma_2\vdash \Delta}{\Gamma_1\Gamma\Gamma_2\vdash \Delta_1\Delta\Delta_2} \qquad \begin{array}{l}\text{provided } \Delta_2\Gamma_2 \text{ or } \Delta_1\Gamma_1\\ \text{or } \Delta_1\Delta_2 \text{ or } \Gamma_1\Gamma_2 \text{ empty.}\end{array}$$

Decomposed into single rules (Cut)$_{\mathcal{B}}$ contains

$$(\mathrm{Cut.1})_{\mathcal{B}}\ \dfrac{\Gamma\vdash \Delta_1 A \quad \Gamma_1 A\vdash \Delta}{\Gamma_1\Gamma\vdash \Delta_1\Delta} \qquad\qquad (\mathrm{Cut.2})_{\mathcal{B}}\ \dfrac{\Gamma\vdash A\Delta_2 \quad A\Gamma_2\vdash \Delta}{\Gamma\Gamma_2\vdash \Delta\Delta_2}$$

as well as (Cut.3) and (Cut.4), which are the same in the $\mathcal{A}$- and $\mathcal{B}$-systems.

Whereas in the critical cases of (Cut)$_{\mathcal{A}}$, A occurs in opposite outermost positions (rightmost in succedent of left premiss/leftmost in antecedent of right premiss, or leftmost in succedent of left premiss/rightmost in antecedent of right premiss), in (Cut)$_{\mathcal{B}}$ is occurs in parallel outermost positions (rightmost/rightmost or leftmost/leftmost).

With (Cut)$_{\mathcal{B}}$ the negation axioms (N1) and (N4) are compatible, whereas with (N2) and (N3) examples of exchange can be derived.

Further candidates of a cut rule which might be generated by changing the critical cases in other ways (but, of course, by keeping (Cut.3) and (Cut.4)) can be excluded since they are incompatible with *any* pair of negation axioms.

We conclude that there are essentially two alternative versions of cut for classical Lambek logic: (Cut)$_{\mathcal{A}}$ and (Cut)$_{\mathcal{B}}$, the first one being compatible with negation axioms (N2) and (N3), and the second one with negation axioms (N1) and (N4).

3. Negation rules

As is well known, in classical (multiple sequent) systems implication and negation are interdefinable (in the presence of certain other connectives). We follow Abrusci [1] in choosing negation as a primitive operation. Lambek [6] uses implication instead.

Corresponding to the two acceptable versions of cut — $(\mathrm{Cut})_{\mathcal{A}}$ and $(\mathrm{Cut})_{\mathcal{B}}$ — there are two possible sets of negation rules. We start with system $\mathcal{A}$.

As we have seen, $(\mathrm{Cut})_{\mathcal{A}}$ is compatible with each pair of negation axioms (N2) and (N3). Obviously we cannot have both pairs with the same negation, because otherwise we would obtain the incompatible pairs of negations (N1) and (N4) as well by taking one axiom from each of (N2) and (N3). This means that we have to use two different negations in order to use both (N2) and (N3). Each one would restrict the most general pair of negation rules

$$\frac{\Gamma_1 A \Gamma_2 \vdash \Delta_1 \Delta_2}{\Gamma_1 \Gamma_2 \vdash \Delta_1 \neg A \Delta_2} \qquad \frac{\Gamma_1 \Gamma_2 \vdash \Delta_1 A \Delta_2}{\Gamma_1 \neg A \Gamma_2 \vdash \Delta_1 \Delta_2}$$

in a different way.

In view of the previous considerations and examples concerning the derivability of exchange, it is clear that both A and $\neg A$ must occur in outermost positions of the antecedents or succedents. For the two axioms of (N2) to be the conclusions of negation rules, there are two choices of right and left introduction rules:

$$(\vdash\neg)_1 \frac{\Gamma A \vdash \Delta}{\Gamma \vdash \Delta \neg A} \quad (\neg\vdash)_1 \frac{\Gamma \vdash A \Delta}{\neg A \Gamma \vdash \Delta} \qquad (\vdash\neg)_2 \frac{A \Gamma \vdash \Delta}{\Gamma \vdash \Delta \neg A} \quad (\neg\vdash)_2 \frac{\Gamma \vdash \Delta A}{\neg A \Gamma \vdash \Delta}\ .$$

Together with the initial axiom $A \vdash A$, each of $(\vdash\neg)_1$ and $(\vdash\neg)_2$ generates $\vdash A \neg A$, and each of $(\neg\vdash)_1$ and $(\neg\vdash)_2$ generates $\neg A A \vdash$. Conversely, the rules $(\vdash\neg)_1$ and $(\neg\vdash)_1$ are derivable from $(\mathrm{Cut})_{\mathcal{A}}$ together with $\vdash A \neg A$ and $\neg A A \vdash$. This means that $(\vdash\neg)_1$ and $(\neg\vdash)_1$ are always available, even if one chooses $(\vdash\neg)_1/(\neg\vdash)_2$ or $(\vdash\neg)_2/(\neg\vdash)_1$ or $(\vdash\neg)_2/(\neg\vdash)_2$ as a pair of negation rules. Actually, any of the three latter choices generates a form of exchange, as the following example with $(\neg\vdash)_2$ shows:

$$\frac{\vdash B \neg B \qquad \dfrac{\Gamma \vdash \Delta A B}{\neg B \Gamma \vdash \Delta A}}{\Gamma \vdash B \Delta A}\ .$$

Therefore, in combination with $(\mathrm{Cut})_{\mathcal{A}}$, the pair $(\vdash\neg)_1/(\neg\vdash)_1$ represents appropriate negation rules corresponding to (N2).

Similarly, assuming the following rules:

$$(\vdash\neg)_3 \frac{A \Gamma \vdash \Delta}{\Gamma \vdash \neg A \Delta} \quad (\neg\vdash)_3 \frac{\Gamma \vdash \Delta A}{\Gamma \neg A \vdash \Delta} \qquad (\vdash\neg)_4 \frac{\Gamma A \vdash \Delta}{\Gamma \vdash \neg A \Delta} \quad (\neg\vdash)_4 \frac{\Gamma \vdash A \Delta}{\Gamma \neg A \vdash \Delta}$$

we can justify the pair $(\vdash\neg)_3/(\neg\vdash)_3$ as corresponding to (N3).

Adopting the notation ${}^{\perp}(\text{-})$ for the negation corresponding to (N2) and $(\text{-})^{\perp}$ for the negation corresponding to (N3) we obtain the following negation rules in the system based on $(\mathrm{Cut})_{\mathcal{A}}$:

$$(\vdash(\text{-})^{\perp})_{\mathcal{A}} \frac{A \Gamma \vdash \Delta}{\Gamma \vdash A^{\perp} \Delta} \quad ((\text{-})^{\perp}\vdash)_{\mathcal{A}} \frac{\Gamma \vdash \Delta A}{\Gamma A^{\perp} \vdash \Delta} \quad (\vdash{}^{\perp}(\text{-}))_{\mathcal{A}} \frac{\Gamma A \vdash \Delta}{\Gamma \vdash \Delta {}^{\perp}A} \quad ({}^{\perp}(\text{-})\vdash)_{\mathcal{A}} \frac{\Gamma \vdash A \Delta}{{}^{\perp}A \Gamma \vdash \Delta}\ .$$

These are exactly the negation rules proposed by Abrusci [1][2].

[2] To avoid terminological confusions, we also follow Abrusci in the mixture of prefix and postfix notations, although two prefix negations would be easier to handle later on.

If we base our system on $(\mathrm{Cut})_{\mathcal{B}}$, by analogous reasoning we obtain the following negation rules:

$$(\vdash(\text{-})^{\perp})_{\mathcal{B}}\,\frac{\Gamma A\vdash\Delta}{\Gamma\vdash A^{\perp}\Delta}\quad((\text{-})^{\perp}\vdash)_{\mathcal{B}}\,\frac{\Gamma\vdash\Delta A}{A^{\perp}\Gamma\vdash\Delta}\quad(\vdash{}^{\perp}(\text{-}))_{\mathcal{B}}\,\frac{A\Gamma\vdash\Delta}{\Gamma\vdash\Delta{}^{\perp}A}\quad({}^{\perp}(\text{-})\vdash)_{\mathcal{B}}\,\frac{\Gamma\vdash A\Delta}{\Gamma{}^{\perp}A\vdash\Delta}\ .$$

4. Binary multiplicatives

Rules for multiplicative conjunction (times, $\otimes$) and disjunction (par, $⅋$) can be justified by treating those connectives as structure connectives, i.e. connectives expressing the structural association of formulas. Considering $\otimes$ as a structure connective for the antecedent and $⅋$ for the succedent, we arrive at the following rules:

$$(\otimes\vdash)\ \frac{\Gamma_1 AB\Gamma_2\vdash\Delta}{\Gamma_1 A\otimes B\Gamma_2\vdash\Delta}\qquad(\otimes\mathrm{E})\ \frac{\Gamma_1 A\otimes B\Gamma_2\vdash\Delta}{\Gamma_1 AB\Gamma_2\vdash\Delta}$$

$$(\vdash⅋)\ \frac{\Gamma\vdash\Delta_1 AB\Delta_2}{\Gamma\vdash\Delta_1 A⅋B\Delta_2}\qquad(⅋\mathrm{E})\ \frac{\Gamma\vdash\Delta_1 A⅋B\Delta_2}{\Gamma\vdash\Delta_1 AB\Delta_2}$$

The rules $(\otimes\vdash)$ and $(\vdash⅋)$ are sequent-style Left- and Right-introduction rules, respectively. $(\otimes\mathrm{E})$ and $(⅋\mathrm{E})$ are kinds of elimination rules which do not fit into the sequent-calculus pattern. However, in the presence of the initial axiom $A\vdash A$ and the uncritical cases of cut, $(\otimes\mathrm{E})$ and $(⅋\mathrm{E})$ are interderivable with the axioms

$$(\otimes\text{ axiom})\quad AB\vdash A\otimes B\qquad\qquad(⅋\text{ axiom})\quad A⅋B\vdash AB\ ,$$

respectively. In the presence of $(\mathrm{Cut})_{\mathcal{A}}$, ($\otimes$ axiom) and ($⅋$ axiom) are interderivable with the rules

$$(\vdash\otimes.1)_{\mathcal{A}}\ \frac{\Gamma_1\vdash\Delta_1 A\quad\Gamma_2\vdash B\Delta_4}{\Gamma_1\Gamma_2\vdash\Delta_1 A\otimes B\Delta_4}\qquad(⅋\vdash.1)_{\mathcal{A}}\ \frac{\Gamma_1 A\vdash\Delta_1\quad B\Gamma_4\vdash\Delta_2}{\Gamma_1 A⅋B\Gamma_4\vdash\Delta_1\Delta_2}\ ,$$

respectively. Both rules are sequent-style rules in the genuine sense. The rule $(\vdash\otimes.1)_{\mathcal{A}}$ is the Right-introduction rule for $\otimes$ proposed by Lambek [6, p. 219]. To these rules we add the following rules, which in the presence of $(\mathrm{Cut})_{\mathcal{A}}$ do not add any power (they are easily derivable from ($\otimes$ axiom) and ($⅋$ axiom)), but are necessary for cut elimination (see Section 1 of Part II):

$$(\vdash\otimes.2)_{\mathcal{A}}\ \frac{\vdash\Delta_1 A\quad\Gamma_2\vdash\Delta_3 B\Delta_4}{\Gamma_2\vdash\Delta_3\Delta_1 A\otimes B\Delta_4}\qquad(⅋\vdash.2)_{\mathcal{A}}\ \frac{\Gamma_1 A\vdash\quad\Gamma_3 B\Gamma_4\vdash\Delta_2}{\Gamma_3\Gamma_1 A⅋B\Gamma_4\vdash\Delta_2}$$

$$(\vdash\otimes.3)_{\mathcal{A}}\ \frac{\Gamma_1\vdash\Delta_1 A\Delta_2\quad\vdash B\Delta_4}{\Gamma_1\vdash\Delta_1 A\otimes B\Delta_4\Delta_2}\qquad(⅋\vdash.3)_{\mathcal{A}}\ \frac{\Gamma_1 A\Gamma_2\vdash\Delta_1\quad B\Gamma_4\vdash}{\Gamma_1 A⅋B\Gamma_4\Gamma_2\vdash\Delta_1}$$

Thus we obtain the following concise form of the $(\vdash\otimes)$ and $(⅋\vdash)$ rules in the system based on $(\mathrm{Cut})_{\mathcal{A}}$:

$$(\vdash\otimes)_{\mathcal{A}}\ \frac{\Gamma_1\vdash\Delta_1 A\Delta_2\quad\Gamma_2\vdash\Delta_3 B\Delta_4}{\Gamma_1\Gamma_2\vdash\Delta_3\Delta_1 A\otimes B\Delta_4\Delta_2}\qquad(⅋\vdash)_{\mathcal{A}}\ \frac{\Gamma_1 A\Gamma_2\vdash\Delta_1\quad\Gamma_3 B\Gamma_4\vdash\Delta_2}{\Gamma_3\Gamma_1 A⅋B\Gamma_4\Gamma_2\vdash\Delta_1\Delta_2}$$

provided $\Delta_2\Delta_3$ or $\Delta_2\Gamma_1$ or $\Delta_3\Gamma_2$ empty; provided $\Gamma_2\Gamma_3$ or $\Gamma_2\Delta_1$ or $\Gamma_3\Delta_2$ empty.

The rules $(\otimes\vdash)$ and $(\vdash\parr)$ are independent of the form of cut. The rules for $\otimes$ and $\parr$ correspond to those proposed by Abrusci [1].

In the system based on $(\mathrm{Cut})_{\mathcal{B}}$, we argue analogously, $(\vdash\otimes)_{\mathcal{B}}$ and $(\parr\vdash)_{\mathcal{B}}$ now being the equivalents of ($\otimes$ axiom) and ($\parr$ axiom) in the presence of $(\mathrm{Cut})_{\mathcal{B}}$ rather than $(\mathrm{Cut})_{\mathcal{A}}$. We obtain the following rules:

$$(\vdash\otimes)_{\mathcal{B}}\ \frac{\Gamma_1\vdash\Delta_1 A\Delta_2 \quad \Gamma_2\vdash\Delta_3 B\Delta_4}{\Gamma_1\Gamma_2\vdash\Delta_1\Delta_3 A\otimes B\Delta_2\Delta_4} \qquad (\parr\vdash)_{\mathcal{B}}\ \frac{\Gamma_1 A\Gamma_2\vdash\Delta_1 \quad \Gamma_3 B\Gamma_4\vdash\Delta_2}{\Gamma_1\Gamma_3 A\parr B\Gamma_2\Gamma_4\vdash\Delta_1\Delta_2}$$

provided $\Delta_1\Delta_4$ or $\Delta_4\Gamma_2$ or $\Delta_1\Gamma_1$ empty; provided $\Gamma_1\Gamma_4$ or $\Gamma_4\Delta_2$ or $\Gamma_1\Delta_1$ empty.

The rules $(\vdash\otimes)$ and $(\parr\vdash)$ are the same in systems $\mathcal{A}$ and $\mathcal{B}$. Again, in the provisos the first disjunct is the crucial one — the others are added only for the purpose of cut elimination.

The remaining connectives: constants and binary additives are unproblematic since they are not structure-sensitive. The formulation of their rules is obvious and independent of whether $(\mathrm{Cut})_{\mathcal{A}}$ or $(\mathrm{Cut})_{\mathcal{B}}$ is chosen as the starting point (see Table 1).

The following result shows that the two basic systems $\mathcal{A}$ and $\mathcal{B}$ for classical Lambek logic, which we have motivated, can easily be translated into each other.

Let $\overline{\Gamma}$ be the reversal of Γ, i.e., if Γ is $A_1 \dots A_n$, then $\overline{\Gamma}$ is $A_n \dots A_1$. Let the "left side reversal" and "right side reversal" of a rule

$$\frac{\Gamma_1\vdash\Delta_1 \quad \Gamma_2\vdash\Delta_2}{\Gamma_3\vdash\Delta_3}$$

be the rules

$$\frac{\overline{\Gamma_1}\vdash\Delta_1 \quad \overline{\Gamma_2}\vdash\Delta_2}{\overline{\Gamma_3}\vdash\Delta_3} \qquad \frac{\Gamma_1\vdash\overline{\Delta_1} \quad \Gamma_2\vdash\overline{\Delta_2}}{\Gamma_3\vdash\overline{\Delta_3}},$$

respectively, and similarly for one-premiss rules. (Provisos are not affected by this reversal, since they deal only with lists of formulas being empty or not.) Let $(\vdash\overline{\otimes})_{\mathcal{A}}$ be the rule obtained from $(\vdash\otimes)_{\mathcal{A}}$ by replacing $A\otimes B$ with $B\otimes A$ in the conclusion, and analogously for all $\otimes$- and $\parr$-rules in $\mathcal{A}$ and $\mathcal{B}$. For example, $(\vdash\overline{\parr})$ and $(\vdash\overline{\otimes})_{\mathcal{B}}$ are

$$\frac{\Gamma\vdash\Delta_1 AB\Delta_2}{\Gamma\vdash\Delta_1 B\parr A\Delta_2} \qquad \frac{\Gamma_1\vdash\Delta_1 A\Delta_2 \quad \Gamma_2\vdash\Delta_3 B\Delta_4}{\Gamma_1,\Gamma_2\vdash\Delta_1\Delta_3 B\otimes A\Delta_2\Delta_4},$$

respectively ($(\vdash\overline{\otimes})_{\mathcal{B}}$ with the same proviso as $(\vdash\otimes)_{\mathcal{B}}$). For rules $\mathcal{R}$ and $\mathcal{R}'$ the notation

$$\mathcal{R} \overset{\mathbf{L}}{\longleftrightarrow} \mathcal{R}'$$

means that, if an instance of $\mathcal{R}$ is given, then by forming its left side reversal we obtain an instance of $\mathcal{R}'$, and vice versa. Analogously,

$$\mathcal{R} \overset{\mathbf{R}}{\longleftrightarrow} \mathcal{R}'$$

expresses that by forming the right side reversal of an instance of $\mathcal{R}$ we obtain an instance of $\mathcal{R}'$, and vice versa. Then by checking rules we easily prove the following lemma.

Table 1: Overview of systems $\mathcal{A}$ and $\mathcal{B}$.

$$(\mathrm{Cut})_{\mathcal{A}}\ \frac{\Gamma\vdash\Delta_1 A\Delta_2 \quad \Gamma_1 A\Gamma_2\vdash\Delta}{\Gamma_1\Gamma\Gamma_2\vdash\Delta_1\Delta\Delta_2}$$
provided $\Delta_2\Gamma_1$ or $\Delta_1\Gamma_2$ or $\Delta_1\Delta_2$ or $\Gamma_1\Gamma_2$ empty

$$(\mathrm{Cut})_{\mathcal{B}}\ \frac{\Gamma\vdash\Delta_1 A\Delta_2 \quad \Gamma_1 A\Gamma_2\vdash\Delta}{\Gamma_1\Gamma\Gamma_2\vdash\Delta_1\Delta\Delta_2}$$
provided $\Delta_2\Gamma_2$ or $\Delta_1\Gamma_1$ or $\Delta_1\Delta_2$ or $\Gamma_1\Gamma_2$ empty

$$(\vdash(\text{-})^{\perp})_{\mathcal{A}}\ \frac{A\Gamma\vdash\Delta}{\Gamma\vdash A^{\perp}\Delta} \quad ((\text{-})^{\perp}\vdash)_{\mathcal{A}}\ \frac{\Gamma\vdash\Delta A}{\Gamma A^{\perp}\vdash\Delta}$$
$$(\vdash{}^{\perp}(\text{-}))_{\mathcal{A}}\ \frac{\Gamma A\vdash\Delta}{\Gamma\vdash\Delta{}^{\perp}A} \quad ({}^{\perp}(\text{-})\vdash)_{\mathcal{A}}\ \frac{\Gamma\vdash A\Delta}{{}^{\perp}A\Gamma\vdash\Delta}$$

$$(\vdash(\text{-})^{\perp})_{\mathcal{B}}\ \frac{\Gamma A\vdash\Delta}{\Gamma\vdash A^{\perp}\Delta} \quad ((\text{-})^{\perp}\vdash)_{\mathcal{B}}\ \frac{\Gamma\vdash\Delta A}{A^{\perp}\Gamma\vdash\Delta}$$
$$(\vdash{}^{\perp}(\text{-}))_{\mathcal{B}}\ \frac{A\Gamma\vdash\Delta}{\Gamma\vdash\Delta{}^{\perp}A} \quad ({}^{\perp}(\text{-})\vdash)_{\mathcal{B}}\ \frac{\Gamma\vdash A\Delta}{\Gamma^{\perp}A\vdash\Delta}$$

$$(\vdash\otimes)_{\mathcal{A}}\ \frac{\Gamma_1\vdash\Delta_1 A\Delta_2 \quad \Gamma_2\vdash\Delta_3 B\Delta_4}{\Gamma_1\Gamma_2\vdash\Delta_3\Delta_1 A\otimes B\Delta_4\Delta_2}$$
provided $\Delta_2\Delta_3$ or $\Delta_2\Gamma_1$ or $\Delta_3\Gamma_2$ empty

$$(\vdash\otimes)_{\mathcal{B}}\ \frac{\Gamma_1\vdash\Delta_1 A\Delta_2 \quad \Gamma_2\vdash\Delta_3 B\Delta_4}{\Gamma_1\Gamma_2\vdash\Delta_1\Delta_3 A\otimes B\Delta_2\Delta_4}$$
provided $\Delta_1\Delta_4$ or $\Delta_4\Gamma_2$ or $\Delta_1\Gamma_1$ empty

$$(⅋\vdash)_{\mathcal{A}}\ \frac{\Gamma_1 A\Gamma_2\vdash\Delta_1 \quad \Gamma_3 B\Gamma_4\vdash\Delta_2}{\Gamma_3\Gamma_1 A⅋B\Gamma_4\Gamma_2\vdash\Delta_1\Delta_2}$$
provided $\Gamma_2\Gamma_3$ or $\Gamma_2\Delta_1$ or $\Gamma_3\Delta_2$ empty

$$(⅋\vdash)_{\mathcal{B}}\ \frac{\Gamma_1 A\Gamma_2\vdash\Delta_1 \quad \Gamma_3 B\Gamma_4\vdash\Delta_2}{\Gamma_1\Gamma_3 A⅋B\Gamma_2\Gamma_4\vdash\Delta_1\Delta_2}$$
provided $\Gamma_1\Gamma_4$ or $\Gamma_4\Delta_2$ or $\Gamma_1\Delta_1$ empty

Axioms and rules common to $\mathcal{A}$ and $\mathcal{B}$

Initial axiom: $A\vdash A$

$(\vdash\mathbf{1})\ \vdash\mathbf{1} \quad (\vdash\top)\ \Gamma\vdash\Delta_1\top\Delta_2 \quad (\mathbf{0}\vdash)\ \mathbf{0}\vdash \quad (\bot\vdash)\ \Gamma_1\bot\Gamma_2\vdash\Delta$

$$(\vdash\mathbf{0})\ \frac{\Gamma\vdash\Delta_1\Delta_2}{\Gamma\vdash\Delta_1\mathbf{0}\Delta_2} \qquad (\mathbf{1}\vdash)\ \frac{\Gamma_1\Gamma_2\vdash\Delta}{\Gamma_1\mathbf{1}\Gamma_2\vdash\Delta}$$

$$(\vdash⅋)\ \frac{\Gamma\vdash\Delta_1 AB\Delta_2}{\Gamma\vdash\Delta_1 A⅋B\Delta_2} \qquad (\otimes\vdash)\ \frac{\Gamma_1 AB\Gamma_2\vdash\Delta}{\Gamma_1 A\otimes B\Gamma_2\vdash\Delta}$$

$$(\vdash\&)\ \frac{\Gamma\vdash\Delta_1 A\Delta_2 \quad \Gamma\vdash\Delta_1 B\Delta_2}{\Gamma\vdash\Delta_1 A\&B\Delta_2} \qquad (\&\vdash)\ \frac{\Gamma_1 A\Gamma_2\vdash\Delta}{\Gamma_1 A\&B\Gamma_2\vdash\Delta}\ \frac{\Gamma_1 B\Gamma_2\vdash\Delta}{\Gamma_1 A\&B\Gamma_2\vdash\Delta}$$

$$(\vdash\oplus)\ \frac{\Gamma\vdash\Delta_1 A\Delta_2}{\Gamma\vdash\Delta_1 A\oplus B\Delta_2}\ \frac{\Gamma\vdash\Delta_1 B\Delta_2}{\Gamma\vdash\Delta_1 A\oplus B\Delta_2} \qquad (\oplus\vdash)\ \frac{\Gamma_1 A\Gamma_2\vdash\Delta \quad \Gamma_1 B\Gamma_2\vdash\Delta}{\Gamma_1 A\oplus B\Gamma_2\vdash\Delta}$$

System $\mathcal{A}$ corresponds to Abrusci's [1] calculus PNCL, extended with $\mathbf{0}$ and $\bot$.

Lemma 1. *For the rules of the systems $\mathcal{A}$ and $\mathcal{B}$ we have the following:*

$$
\begin{array}{ccc}
(Cut)_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} (Cut)_{\mathcal{B}} & \qquad & (Cut)_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} (Cut)_{\mathcal{B}} \\
(\vdash(\text{-})^{\perp})_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} (\vdash(\text{-})^{\perp})_{\mathcal{B}} & & (\vdash(\text{-})^{\perp})_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} (\vdash{}^{\perp}(\text{-}))_{\mathcal{B}} \\
((\text{-})^{\perp}\vdash)_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} ((\text{-})^{\perp}\vdash)_{\mathcal{B}} & & ((\text{-})^{\perp}\vdash)_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} ({}^{\perp}(\text{-})\vdash)_{\mathcal{B}} \\
(\vdash{}^{\perp}(\text{-}))_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} (\vdash{}^{\perp}(\text{-}))_{\mathcal{B}} & & (\vdash{}^{\perp}(\text{-}))_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} (\vdash(\text{-})^{\perp})_{\mathcal{B}} \\
({}^{\perp}(\text{-})\vdash)_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} ({}^{\perp}(\text{-})\vdash)_{\mathcal{B}} & & ({}^{\perp}(\text{-})\vdash)_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} ((\text{-})^{\perp}\vdash)_{\mathcal{B}} \\
(\vdash\otimes)_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} (\vdash\overline{\otimes})_{\mathcal{B}} & & (\vdash\otimes)_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} (\vdash\otimes)_{\mathcal{B}} \\
(\otimes\vdash)_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} (\overline{\otimes}\vdash)_{\mathcal{B}} & & (\otimes\vdash)_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} (\otimes\vdash)_{\mathcal{B}} \\
(\vdash\overline{\otimes})_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} (\vdash\otimes)_{\mathcal{B}} & & (\vdash\overline{\otimes})_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} (\vdash\overline{\otimes})_{\mathcal{B}} \\
(\overline{\otimes}\vdash)_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} (\otimes\vdash)_{\mathcal{B}} & & (\overline{\otimes}\vdash)_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} (\overline{\otimes}\vdash)_{\mathcal{B}} \\
(\vdash\parr)_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} (\vdash\parr)_{\mathcal{B}} & & (\vdash\parr)_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} (\vdash\overline{\parr})_{\mathcal{B}} \\
(\parr\vdash)_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} (\parr\vdash)_{\mathcal{B}} & & (\parr\vdash)_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} (\overline{\parr}\vdash)_{\mathcal{B}} \\
(\vdash\overline{\parr})_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} (\vdash\overline{\parr})_{\mathcal{B}} & & (\vdash\overline{\parr})_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} (\vdash\parr)_{\mathcal{B}} \\
(\overline{\parr}\vdash)_{\mathcal{A}} \stackrel{\mathbf{L}}{\longleftrightarrow} (\overline{\parr}\vdash)_{\mathcal{B}} & & (\overline{\parr}\vdash)_{\mathcal{A}} \stackrel{\mathbf{R}}{\longleftrightarrow} (\parr\vdash)_{\mathcal{B}} \\
\mathcal{R} \stackrel{\mathbf{L}}{\longleftrightarrow} \mathcal{R} & & \mathcal{R} \stackrel{\mathbf{R}}{\longleftrightarrow} \mathcal{R}
\end{array}
$$

for any other rule $\mathcal{R}$.

From that we obtain the following embedding between systems.

Theorem 2. *Let Γ' be obtained from Γ by recursively replacing in every formula of Γ every subformula of the form $A \otimes B$ with $B \otimes A$. Let Γ'' be obtained from Γ by recursively replacing in every formula of Γ every subformula of the form $A \parr B$ with $B \parr A$, and by interchanging $(\text{-})^{\perp}$ and ${}^{\perp}(\text{-})$. Then the following holds:*

$$(1)\ \Gamma \vdash_{\mathcal{A}} \Delta \ \textit{iff}\ \overline{\Gamma'} \vdash_{\mathcal{B}} \Delta' \qquad (2)\ \Gamma \vdash_{\mathcal{B}} \Delta \ \textit{iff}\ \Gamma'' \vdash_{\mathcal{A}} \overline{\Delta''}$$

Obviously, if Γ^{+} is $(\Gamma')''$, then we have as a corollary:

$$\Gamma \vdash_{\mathcal{A}} \Delta \ \textit{iff}\ \overline{\Gamma^{+}} \vdash_{\mathcal{A}} \overline{\Delta^{+}} \qquad \Gamma \vdash_{\mathcal{B}} \Delta \ \textit{iff}\ \overline{\Gamma^{+}} \vdash_{\mathcal{B}} \overline{\Delta^{+}}$$

Theorem 2 justifies that we restrict ourselves to system $\mathcal{A}$.

Part II: Cut elimination

Since we can rely on system $\mathcal{A}$, in the notation of rules we will skip the index $\mathcal{A}$.

1. Cut elimination for positive logic

For the system without negation, the cut rule can be eliminated. We just sketch the essential points, focussing on the connective $\otimes$. The *main reductions* ("logical reductions"), which take place when the cut formula of a topmost cut is introduced into both premisses of the cut in the last step, are obvious. For example,

$$\frac{\dfrac{\Gamma_1 \vdash \Delta_1 A \quad \Gamma_2 \vdash B \Delta_4}{\Gamma_1 \Gamma_2 \vdash \Delta_1 A \otimes B \Delta_4} \quad \dfrac{AB \vdash \Delta}{A \otimes B \vdash \Delta}}{\Gamma_1 \Gamma_2 \vdash \Delta_1 \Delta \Delta_4}$$

is reduced to

$$\frac{\Gamma_2 \vdash B \Delta_4 \quad \dfrac{\Gamma_1 \vdash \Delta_1 A \quad AB \vdash \Delta}{\Gamma_1 B \vdash \Delta_1 \Delta}}{\Gamma_1 \Gamma_2 \vdash \Delta_1 \Delta \Delta_4} \quad ,$$

and similarly for all other cases of main reductions. For the *permutative reductions*, by means of which a topmost cut and a logical inference are interchanged, it must be made sure that the permutations are compatible with the provisos of $(\vdash\otimes)$ and (Cut). For example,

$$\frac{\Gamma_1 \vdash \Delta_3 C \quad \dfrac{\vdash \Delta_1 A \quad C \Gamma_2 \vdash B \Delta_2}{C \Gamma_2 \vdash \Delta_1 A \otimes B \Delta_2}\,(\vdash \otimes .1)}{\Gamma_1 \Gamma_2 \vdash \Delta_3 \Delta_1 A \otimes B \Delta_2}\,(\mathrm{Cut}.1)$$

is reduced to

$$\frac{\vdash \Delta_1 A \quad \dfrac{\Gamma_1 \vdash \Delta_3 C \quad C \Gamma_2 \vdash B \Delta_2}{\Gamma_1 \Gamma_2 \vdash \Delta_3 B \Delta_2}\,(\mathrm{Cut}.1)}{\Gamma_1 \Gamma_2 \vdash \Delta_3 \Delta_1 A \otimes B \Delta_2}\,(\vdash \otimes .2)$$

The application of $(\vdash \otimes .2)$ in the second derivation is not at the same time one of $(\vdash \otimes .1)$. This shows that it is essential to have $(\vdash \otimes .2)$ included in $(\vdash\otimes)$ in addition to $(\vdash \otimes .1)$, although it can be derived from $(\vdash \otimes .1)$ by using cut.

2. The failure of cut elimination in the full system

For atomic A, the sequent $({}^{\perp\perp}A)^{\perp\perp} \vdash A$ is derivable with, but not without cut:

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A, {}^{\perp}A} \quad \dfrac{\dfrac{{}^{\perp}A \vdash {}^{\perp}A}{{}^{\perp\perp}A, {}^{\perp}A \vdash}}{{}^{\perp}A \vdash ({}^{\perp\perp}A)^{\perp}}}{\vdash A, ({}^{\perp\perp}A)^{\perp}}\,(\mathrm{Cut})}{({}^{\perp\perp}A)^{\perp\perp} \vdash A} \qquad\qquad \dfrac{\dfrac{?}{\vdash A, ({}^{\perp\perp}A)^{\perp}}\,(\text{Non-Cut})}{({}^{\perp\perp}A)^{\perp\perp} \vdash A}$$

since there is no inference apart from cut which can generate $\vdash A, ({}^{\perp\perp}A)^{\perp}$.[3]

[3] This example is due to Abrusci [1, p. 1449f.]. It is strange that Abrusci does not read it as a decisive proof of the non-admissibility of cut but just of the fact that cut is not eliminable "by the usual cut elimination procedures". Lambek [6] erroneously claims that cut is eliminable.

The reason for this failure of cut elimination in the full system is that the negation rules prevent appropriate permutative reductions. For example, in the derivation

$$\dfrac{\Gamma_1 \vdash \Delta_1 B \qquad \dfrac{A B \Gamma_2 \vdash \Delta_2}{B \Gamma_2 \vdash A^{\perp} \Delta_2}\,(\vdash(\text{-})^{\perp})}{\Gamma_1 \Gamma_2 \vdash \Delta_1 A^{\perp} \Delta_2}\,(\text{Cut})$$

(Cut) cannot be moved upwards and interchanged with ($\vdash$(-)$^{\perp}$), since $\Gamma_1 \vdash \Delta_1 B$ and $A B \Gamma_2 \vdash \Delta_2$ are not suitable as premisses of (Cut): According to the provisos, the A in front of B blocks the cut with B. Obviously, this problem is due to formulas changing sides, which is inherent in the negation rules.

It is interesting that for a Tait-style one-sided sequent calculus, into which $\mathcal{A}$ can be translated using cut, cut elimination can be proved (see [1, 7]). This shows that a one-sided system is not always just an economic variant (with the number of inference rules being halved) of the corresponding two-sided system, as often claimed in the literature, but may behave differently.

The following counterexample to the admissibility of cut in $\mathcal{A}$ shows that nested negations or the usage of two different negations are not essential. Consider the following derivation, with A, B, C, D, E, F, G being atomic:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A \parr B \vdash A, B} \quad C \vdash C}{(A \parr B) \parr C \vdash A, B, C} \quad D \vdash D}{((A \parr B) \parr C) \parr D \vdash A, B, C, D}}{((A \parr B) \parr C) \parr D \vdash A, B, C \parr D}}{((A \parr B) \parr C) \parr D, (C \parr D)^{\perp} \vdash A, B} \quad \dfrac{\dfrac{\dfrac{E \vdash E \quad \dfrac{F \vdash F \quad \dfrac{B \vdash B \quad G \vdash G}{B, G \vdash B \otimes G}}{F, B, G \vdash F \otimes (B \otimes G)}}{E, F, B, G \vdash E \otimes (F \otimes (B \otimes G))}}{E \otimes F, B, G \vdash E \otimes (F \otimes (B \otimes G))}}{B, G \vdash (E \otimes F)^{\perp}, E \otimes (F \otimes (B \otimes G))}}{((A \parr B) \parr C) \parr D, (C \parr D)^{\perp}, G \vdash A, (E \otimes F)^{\perp}, E \otimes (F \otimes (B \otimes G))} \quad .$$

To show that there is no cut-free derivation of the end sequent, we use the following necessary condition for a sequent $\Gamma \vdash \Delta$ to be derivable without cut: Any atomic formula occurring as a subformula in $\Gamma \vdash \Delta$ has to occur at least twice. This is due to contraction rules being absent.

If in a cut-free derivation of $((A \parr B) \parr C) \parr D, (C \parr D)^{\perp}, G \vdash A, (E \otimes F)^{\perp}, E \otimes (F \otimes (B \otimes G))$ the rule ($\vdash\otimes$.1) were applied in the last step, then the right premiss of this application would be $\Gamma \vdash F \otimes (B \otimes G)$, with Γ as some part of $((A \parr B) \parr C) \parr D, (C \parr D)^{\perp}, G$. Thus F would occur only once. If ($\vdash \otimes$ .2) were applied in the last step, then the left premiss of this application would be either $\vdash A, (E \otimes F)^{\perp}, E$ or $\vdash (E \otimes F)^{\perp}, E$ or $\vdash E$, so either E or F would occur only once. An application of ($\vdash \otimes$ .3) would be one of ($\vdash \otimes$ .1) at the same time. For possible applications of ($\parr\vdash$) one argues analogously. Besides ($\vdash\otimes$) and ($\parr\vdash$) no other rule can have been applied in the last step.

The last example also shows that without cut the associativity of $\otimes$ and $\parr$ is only admissible, but not derivable. For example, if the rule

$$\dfrac{\Gamma \vdash \Delta_1 (A \otimes B) \otimes C \Delta_2}{\Gamma \vdash \Delta_1 A \otimes (B \otimes C) \Delta_2}$$

were derivable without cut, then the end sequent of the above example derivation could be obtained from the sequent $(A \parr B) \parr (C \parr D), (C \parr D)^{\perp}, G \vdash A, (E \otimes F)^{\perp}, (E \otimes F) \otimes (B \otimes G)$, which can be derived without cut using $(\vdash \otimes.2)$ and $(\parr \vdash.3)$. That associativity is admissible follows from the cut elimination result for the positive system and the derivability without cut of $A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C$.

However, for the counterexamples to work it is essential that in the scope of a negation sign some other connective occurs, as will be demonstrated in the next section.

3. Restricted cut elimination

We show that cut is eliminable if we restrict ourselves to derivations of simple sequents. A sequent $\Gamma \vdash \Delta$ is called simple if it consists of simple formulas only. A formula is called simple if in the scope of a negation no logical sign except the same negation is allowed to occur. In other words, if $A^{\perp}$ is a subformula of a simple formula, then A is atomic or of the form $B^{\perp \cdots \perp}$ for atomic B, and similarly for ${}^{\perp}\!A$.

Our proof proceeds as follows: We define a system $\mathcal{A}^*$ which is an extension of $\mathcal{A}$. For $\mathcal{A}^*$ we can prove cut elimination, based on certain permutability features of negation rules. Cut-free derivations of simple sequents in $\mathcal{A}^*$ are cut-free derivations in $\mathcal{A}$.

For certain negated formulas C we define C^* as follows:

$$
\begin{array}{rlrl}
(\mathbf{1}^{\perp})^* &= \mathbf{0} & ({}^{\perp}\mathbf{1})^* &= \mathbf{0} \\
(\mathbf{0}^{\perp})^* &= \mathbf{1} & ({}^{\perp}\mathbf{0})^* &= \mathbf{1} \\
(\top^{\perp})^* &= \bot & ({}^{\perp}\top)^* &= \bot \\
(\bot^{\perp})^* &= \top & ({}^{\perp}\bot)^* &= \top \\
(({}^{\perp}\!A)^{\perp})^* &= A & ({}^{\perp}(A^{\perp}))^* &= A \\
((A \otimes B)^{\perp})^* &= B^{\perp} \parr A^{\perp} & ({}^{\perp}(A \otimes B))^* &= {}^{\perp}B \parr {}^{\perp}\!A \\
((A \parr B)^{\perp})^* &= B^{\perp} \otimes A^{\perp} & ({}^{\perp}(A \parr B))^* &= {}^{\perp}B \otimes {}^{\perp}\!A \\
((A \& B)^{\perp})^* &= A^{\perp} \oplus B^{\perp} & ({}^{\perp}(A \& B))^* &= {}^{\perp}\!A \oplus {}^{\perp}B \\
((A \oplus B)^{\perp})^* &= A^{\perp} \& B^{\perp} & ({}^{\perp}(A \oplus B))^* &= {}^{\perp}\!A \& {}^{\perp}B\,.
\end{array}
$$

For formulas C of a different form, C^* is not defined. In particular, $(A^{\perp})^*$ for atomic A and $(A^{\perp\perp})^*$ are not defined.

For formulas A the degree $deg^*(A)$ is defined as follows:
Let $w(A^{\perp})$ be the number of occurrences of logical signs different from $(\text{-})^{\perp}$ in A and $w({}^{\perp}\!A)$ the number of occurrences of logical signs different from ${}^{\perp}(\text{-})$ in A. (For example, if A and B are atomic, $({}^{\perp}(({}^{\perp}\!A \otimes B^{\perp\perp}) \wedge \mathbf{1}))^{\perp\perp}$ is 5 and ${}^{\perp}(({}^{\perp}(({}^{\perp}\!A \otimes B^{\perp\perp}) \wedge \mathbf{1}))^{\perp})$ is 6.) Then

$$
\begin{array}{l}
deg^*(A) := 0 \quad \text{if } A \text{ is atomic} \\
deg^*(\top) = deg^*(\bot) = deg^*(\mathbf{1}) = deg^*(\mathbf{0}) := 1 \\
deg^*(A \circ B) := max(deg^*(A), deg^*(A)) + 1 \quad \text{for binary connectives } \circ \\
deg^*(A^{\perp}) := deg^*(A) + 1 + w(A^{\perp}) \\
deg^*({}^{\perp}\!A) := deg^*(A) + 1 + w({}^{\perp}\!A)
\end{array}
$$

It is easy to see that $deg^*(A^*) < deg^*(A)$ for any formula A, for which A^* is defined.

In the following the notation $A^{\perp \cdots \perp}$, which expresses that a sequence of negations $(\text{-})^{\perp}$ is applied to A, will be understood as including the empty sequence, in which case $A^{\perp \cdots \perp}$ is A itself, and analogously for ${}^{\perp \cdots \perp}\!A$.

The system $\mathcal{A}^*$ is obtained from the system $\mathcal{A}$ by adding the following rules:

$$(\vdash *)\ \frac{\Gamma\vdash\Delta_1(A^*)^{\perp\cdots\perp}\Delta_2}{\Gamma\vdash\Delta_1 A^{\perp\cdots\perp}\Delta_2}\qquad\qquad \frac{\Gamma\vdash\Delta_1{}^{\perp\cdots\perp}(A^*)\Delta_2}{\Gamma\vdash\Delta_1{}^{\perp\cdots\perp}A\Delta_2}$$

$$(*\vdash)\ \frac{\Gamma_1(A^*)^{\perp\cdots\perp}\Gamma_2\vdash\Delta}{\Gamma_1 A^{\perp\cdots\perp}\Gamma_2\vdash\Delta}\qquad\qquad \frac{\Gamma_1{}^{\perp\cdots\perp}(A^*)\Gamma_2\vdash\Delta}{\Gamma_1{}^{\perp\cdots\perp}A\Gamma_2\vdash\Delta}$$

and by restricting the negation rules in such a way that the formula introduced must always be simple:

$$(\vdash(\text{-})^{\perp})_0\ \frac{A^{\perp\cdots\perp}\Gamma\vdash\Delta}{\Gamma\vdash(A^{\perp\cdots\perp})^{\perp}\Delta}\qquad ((\text{-})^{\perp}\vdash)_0\ \frac{\Gamma\vdash\Delta A^{\perp\cdots\perp}}{\Gamma(A^{\perp\cdots\perp})^{\perp}\vdash\Delta}$$

$$(\vdash{}^{\perp}(\text{-}))_0\ \frac{\Gamma{}^{\perp\cdots\perp}A\vdash\Delta}{\Gamma\vdash\Delta{}^{\perp}({}^{\perp\cdots\perp}A)}\qquad ({}^{\perp}(\text{-})\vdash)_0\ \frac{\Gamma\vdash{}^{\perp\cdots\perp}A\Delta}{{}^{\perp}({}^{\perp\cdots\perp}A)\Gamma\vdash\Delta}\qquad A \text{ atomic.}$$

Note that in the $*$-rules, $(A^*)^{\perp\cdots\perp}$ is uniquely determined by $A^{\perp\cdots\perp}$, and ${}^{\perp\cdots\perp}(A^*)$ by ${}^{\perp\cdots\perp}A$, since A is the maximal subformula of $A^{\perp\cdots\perp}$ or ${}^{\perp\cdots\perp}A$, respectively, for which A^* is defined.

We first show that $\mathcal{A}^*$ is an extension of $\mathcal{A}$.

Lemma 3. *$\Gamma\vdash_{\mathcal{A}}\Delta$ implies $\Gamma\vdash_{\mathcal{A}^*}\Delta$.*

Proof We have to show that the unrestricted negation rules are admissible in $\mathcal{A}^*$. Due to the presence of cut it suffices to show that the unrestricted negation axioms are derivable in $\mathcal{A}^*$:

$$\vdash A^{\perp}, A \qquad A, {}^{\perp}A\vdash \qquad \vdash A, {}^{\perp}A \qquad {}^{\perp}A, A\vdash .$$

We use induction on $deg^*(A)$ and prove the derivability of all axioms simultaneously.
(i) If A is atomic, we use the restricted negation rules available in $\mathcal{A}^*$.
(ii) If A is $\mathbf{1}$, $\mathbf{0}$, $\top$ or $\bot$, we construct derivations like the following ones:

$$\dfrac{\dfrac{\vdash\mathbf{1}}{\vdash\mathbf{0},\mathbf{1}}\,(\vdash\mathbf{0})}{\vdash\mathbf{1}^{\perp},\mathbf{1}}\,(\vdash *)\qquad\qquad \dfrac{\dfrac{\mathbf{0}\vdash}{\mathbf{0},\mathbf{1}\vdash}\,(\mathbf{1}\vdash)}{{}^{\perp}\mathbf{1},\mathbf{1}\vdash}\,(*\vdash)$$

(iii) If A has the form $B\otimes C$, we obtain the following derivation by using the induction hypothesis for B and C:

$$\dfrac{\dfrac{\dfrac{\vdash B^{\perp}, B \quad \vdash C^{\perp}, C}{\vdash C^{\perp}, B^{\perp}, B\otimes C}\,(\vdash\otimes .2)}{\vdash C^{\perp}\parr B^{\perp}, B\otimes C}\,(\vdash\parr)}{\vdash(B\otimes C)^{\perp}, B\otimes C}\,(\vdash *)$$

We proceed analogously for the other binary connectives and the remaining three negation axioms.
(iv) If A has the form ${}^{\perp}B$, we obtain $\vdash A^{\perp}, A$ by using the induction hypothesis for B:

$$\frac{\vdash B, {}^{\perp}B}{\vdash ({}^{\perp}B)^{\perp}, {}^{\perp}B}\,(\vdash *)\quad .$$

In order to obtain $\vdash A, {}^{\perp}A$, we write ${}^{\perp}B$ as ${}^{\perp \cdots \perp}({}^{\perp}C)$ in such a way that C does not start with ${}^{\perp}(\text{-})$. If C is atomic, we obtain $\vdash {}^{\perp}B, {}^{\perp\perp}B$ by the restricted negation rules available in $\mathcal{A}^*$. Otherwise $({}^{\perp}C)^*$ is defined. Thus by induction hypothesis we have $\vdash {}^{\perp \cdots \perp}(({}^{\perp}C)^*), {}^{\perp}({}^{\perp \cdots \perp}(({}^{\perp}C)^*))$, from which by twofold application of $(\vdash *)$ we obtain $\vdash {}^{\perp \cdots \perp}({}^{\perp}C), {}^{\perp}({}^{\perp \cdots \perp}({}^{\perp}C))$, which is the same as $\vdash {}^{\perp}B, {}^{\perp\perp}B$.
For the two negation axioms for the succedent we argue in exactly the same way.
(v) If A has the form $B^{\perp}$, we argue as in (iv), with $(\text{-})^{\perp}$ and ${}^{\perp}(\text{-})$ interchanged. □

In order to show that $\mathcal{A}^*$ admits cut elimination, we prove the following:

Lemma 4. *Any cut-free derviation in $\mathcal{A}^*$ can be transformed into a cut-free derivation in $\mathcal{A}^*$ of the same sequent such that the following holds: Above an application of a $(\text{-})^{\perp}$-rule only initial axioms and applications of $(\text{-})^{\perp}$-rules occur. Correspondingly, above an application of a ${}^{\perp}(\text{-})$-rule only initial axioms and applications of ${}^{\perp}(\text{-})$-rules occur.*

Proof We show that applications of negation rules can be permuted upwards in cut-free derivations in $\mathcal{A}^*$. We consider $(\vdash(\text{-})^{\perp})_0$ as an example. Suppose in

$$(\vdash(\text{-})^{\perp})_0\,\frac{A\Gamma \vdash \Delta}{\Gamma \vdash A^{\perp}\Delta} \quad \text{where } A \text{ has the form } B^{\perp\cdots\perp} \text{ for atomic } B$$

the premiss $A\Gamma \vdash \Delta$ has not been derived by means of a $(\text{-})^{\perp}$-rule. Then a formula different from A has been introduced in the last step. We proceed according to the pattern of the following two examples:

$$\dfrac{\dfrac{A\Gamma B \vdash \Delta}{A\Gamma \vdash \Delta {}^{\perp}B}\,(\vdash {}^{\perp}(\text{-}))}{\Gamma \vdash A^{\perp}\Delta {}^{\perp}B}\,(\vdash(\text{-})^{\perp}) \quad \text{is reduced to} \quad \dfrac{\dfrac{A\Gamma B \vdash \Delta}{\Gamma B \vdash A^{\perp}\Delta}\,(\vdash(\text{-})^{\perp})}{\Gamma \vdash A^{\perp}\Delta {}^{\perp}B}\,(\vdash {}^{\perp}(\text{-}))\quad ,$$

$$\dfrac{\dfrac{A\Gamma_1 B C \Gamma_2 \vdash \Delta}{A\Gamma_1 B \otimes C \Gamma_2 \vdash \Delta}\,(\otimes\vdash)}{\Gamma_1 B \otimes C \Gamma_2 \vdash A^{\perp}\Delta}\,(\vdash(\text{-})^{\perp}) \quad \text{is reduced to} \quad \dfrac{\dfrac{A\Gamma_1 B C \Gamma_2 \vdash \Delta}{\Gamma_1 B C \Gamma_2 \vdash A^{\perp}\Delta}\,(\vdash(\text{-})^{\perp})}{\Gamma_1 B \otimes C \Gamma_2 \vdash A^{\perp}\Delta}\,(\otimes\vdash)\quad . \square$$

As an immediate consequence of this lemma we obtain the following:

Lemma 5. *Without loss of generality we may assume that in a cut-free derivation in $\mathcal{A}^*$ the premiss of an application of $(\vdash(\text{-})^{\perp})_0$ is either of the form $C \vdash C$ or of the form $C, C^{\perp} \vdash$, that of $((\text{-})^{\perp}\vdash)_0$ either $C \vdash C$ or $\vdash C^{\perp}, C$, that of $(\vdash {}^{\perp}(\text{-}))_0$ either $C \vdash C$ or ${}^{\perp}C, C \vdash$, and that of $({}^{\perp}(\text{-})\vdash)_0$ either $C \vdash C$ or $\vdash C, {}^{\perp}C$.*

Now we can prove our main result.

Theorem 6. (Cut) *is admissible in $\mathcal{A}^*$.*

Proof As in the proof for the positive fragment of $\mathcal{A}$ in Section II.1, we just sketch the basic points. As a measure for the complexity of the cut formula A we use $deg^*(A)$. The additional main reductions for the $(\text{-})^{\perp}$-, ${}^{\perp}(\text{-})$-, and $*$-rules are obvious. For example,

$$\dfrac{\dfrac{\Gamma_1 A \vdash \Delta_1}{\Gamma_1 \vdash \Delta_1\, {}^{\perp}\!A} \quad \dfrac{\Gamma_2 \vdash A \Delta_2}{{}^{\perp}\!A\, \Gamma_2 \vdash \Delta_2}}{\Gamma_1 \Gamma_2 \vdash \Delta_1 \Delta_2} \quad \text{is reduced to} \quad \dfrac{\Gamma_2 \vdash A \Delta_2 \quad \Gamma_1 A \vdash \Delta_1}{\Gamma_1 \Gamma_2 \vdash \Delta_1 \Delta_2}$$

$$\dfrac{\dfrac{\Gamma_1 \vdash \Delta_1 (A^*)^{\perp \dots \perp}}{\Gamma_1 \vdash \Delta_1 A^{\perp \dots \perp}} \quad \dfrac{(A^*)^{\perp \dots \perp} \Gamma_2 \vdash \Delta_2}{A^{\perp \dots \perp} \Gamma_2 \vdash \Delta_2}}{\Gamma_1 \Gamma_2 \vdash \Delta_1 \Delta_2} \quad \text{is reduced to} \quad \dfrac{\Gamma_1 \vdash \Delta_1 (A^*)^{\perp \dots \perp} \quad (A^*)^{\perp \dots \perp} \Gamma_2 \vdash \Delta_2}{\Gamma_1 \Gamma_2 \vdash \Delta_1 \Delta_2}$$

The crucial cases are permutative reductions with negation rules involved. A case like

$$\dfrac{\Gamma_1 \vdash \Delta_1 A \quad \dfrac{B A \Gamma_2 \vdash \Delta_2}{A \Gamma_2 \vdash B^{\perp} \Delta_2}\,(\vdash(\text{-})^{\perp})_0}{\Gamma_1 \Gamma_2 \vdash \Delta_1 B^{\perp} \Delta_2}\,(\text{Cut.1}) \quad ,$$

which blocked cut elimination for $\mathcal{A}$, can now be treated as follows: Since we are considering topmost applications of cut, the derivations above the cut are cut-free. Therefore, due to Lemma 5, we may assume that the premiss of $(\vdash(\text{-})^{\perp})_0$ is either of the form $C \vdash C$ or of the form $C C^{\perp} \vdash$. Only the second form matches $B A \Gamma_2 \vdash \Delta_2$, in which case we have

$$\dfrac{\Gamma_1 \vdash \Delta_1 B^{\perp} \quad \dfrac{B B^{\perp} \vdash}{B^{\perp} \vdash B^{\perp}}}{\Gamma_1 \vdash \Delta_1 B^{\perp}}$$

, which is immediately reduced to $\Gamma_1 \vdash \Delta_1 B^{\perp}$. □

In order to obtain a result for $\mathcal{A}$, we use the following lemma.

Lemma 7. *A cut-free derivation in $\mathcal{A}^*$ of a simple sequent $\Gamma \vdash \Delta$ is a cut-free derivation of $\Gamma \vdash \Delta$ in $\mathcal{A}$.*

Proof Since $\Gamma \vdash \Delta$ is simple, $*$-rules cannot be used in a cut-free derivation, since they always generate non-simple formulas. □

Taking these results together, we obtain the following theorem:

Theorem 8. *Any derivation in $\mathcal{A}$ of a simple sequent can be turned into a cut-free derivation in $\mathcal{A}$ of the same sequent.*

The following corollary is crucial for the justification of the inference rules of $\mathcal{A}$.

Corollary 9. *Exchange is not admissible in $\mathcal{A}$.*

Proof Otherwise we would obtain a cut-free derivation in $\mathcal{A}$ of $A \otimes B \vdash B \otimes A$ for atomic A and B, which is not possible.[4] □

[4] This corollary can also be obtained from Abrusci's [1] translation of $\mathcal{A}$ into a one-sided sequent calculus, for which he proves cut elimination. However, since Abrusci's translation only works in the presence of cut, Theorem 8 is not provable from his results.

Appendix

It is instructive to see what happens in $\mathcal{A}^*$ with the derivation in $\mathcal{A}$ of $({}^{\perp\perp}A)^{\perp\perp}\vdash A$, for which no cut-free derivation in $\mathcal{A}$ exists (see Section II.2). In $\mathcal{A}^*$ there is the following cut-free derivation:

$$\dfrac{\dfrac{\dfrac{A\vdash A}{\vdash A, {}^{\perp}A}}{\vdash A, ({}^{\perp\perp}A)^{\perp}}\,(\vdash *)}{({}^{\perp\perp}A)^{\perp\perp}\vdash A}$$

This is exactly what we arrive at when we perform the reductions sketched in the proofs of Theorem 6 and Lemma 4 starting from the derivation with cut in $\mathcal{A}$.

For the second example of Section II.2 we obtain the following cut-free derivation in $\mathcal{A}^*$:

$$\dfrac{\dfrac{\dfrac{\dfrac{E\vdash E}{E^{\perp}, E\vdash}\quad \dfrac{\dfrac{F\vdash F}{F^{\perp}, F\vdash}\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\vdash A\quad B\vdash B}{A\parr B\vdash A, B}\quad G\vdash G}{A\parr B, G\vdash A, B\otimes G}\quad \dfrac{C\vdash C}{C, C^{\perp}\vdash}}{(A\parr B)\parr C, C^{\perp}, G\vdash A, B\otimes G}\quad \dfrac{D\vdash D}{D, D^{\perp}\vdash}}{((A\parr B)\parr C)\parr D, D^{\perp}, C^{\perp}, G\vdash A, B\otimes G}}{((A\parr B)\parr C)\parr D, D^{\perp}\otimes C^{\perp}, G\vdash A, B\otimes G}\,(*\vdash)}{((A\parr B)\parr C)\parr D, (C\parr D)^{\perp}, G\vdash A, B\otimes G}}{((A\parr B)\parr C)\parr D, (C\parr D)^{\perp}, G\vdash A, F^{\perp}, F\otimes(B\otimes G)}}{((A\parr B)\parr C)\parr D, (C\parr D)^{\perp}, G\vdash A, F^{\perp}, E^{\perp}, E\otimes(F\otimes(B\otimes G))}}{((A\parr B)\parr C)\parr D, (C\parr D)^{\perp}, G\vdash A, F^{\perp}\parr E^{\perp}, E\otimes(F\otimes(B\otimes G))}\,(\vdash *)}{((A\parr B)\parr C)\parr D, (C\parr D)^{\perp}, G\vdash A, (E\otimes F)^{\perp}, E\otimes(F\otimes(B\otimes G))}\quad .$$

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