Chapter 1
Popper on Quantification and Identity

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1.1 Introduction

Karl Popper developed a new approach to mathematical logic with foundational aspirations in the 1940s, which was published in a series of articles between 1946 and 1949. This new system of logic did not have the influence that he had hoped for, despite being original, and despite anticipating problems which were discussed in the logic community only much later. In a previous article (Binder and Piecha, 2017) we explored in technical detail his approach to propositional logic, modal logic and various sub-classical systems like intuitionistic, dual-intuitionistic and minimal logic. A detailed discussion of his theory of quantification (i.e., of first-order logic) has, with the exception of an appendix to an article by Schroeder-Heister (1984), been lacking so far. We first present the main ideas of Popper's approach and the core of the propositional system. We then provide a concise introduction to his theory of quantification and identity, accessible to non-specialists. Popper's theory of quantification underwent significant modifications over the course of his published articles, subsequent corrections to those articles, and in unpublished correspondence with other logicians. We present what we consider to be his most mature view on these matters, taking unpublished material into account.

Popper's approach to logic is original, philosophically interesting, and also severely underappreciated. There are only a few detailed expositions and discussions of Popper's works on logic (cf. Schroeder-Heister, 1984, 2006; Binder and Piecha, 2017). Moreover, Popper's ideas on quantification have not yet received an extensive

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discussion, and in this article we would like to provide one. We first give a brief sketch of the genesis of Popper's ideas on logic in § 1.2. In § 1.3 we introduce the central philosophical ideas of Popper's approach to logic, namely to define logical constants by inferential definitions that are based on a deducibility relation. These ideas are exemplified by inferential definitions of connectives of propositional logic. In § 1.4 we show how Popper intended to extend the propositional system to first-order logic (Popper uses the terms "theory of quantification" or "quantification theory" instead of "first-order logic"). At first, he extends his concept of object language to include open statements and his deducibility relation to range over open statements. He then adds a substitution operation which replaces free variables by other free variables, and gives rules and postulates which characterize this substitution operation. We discuss his definitions of the auxiliary concepts of identity and non-free-occurrence of a variable in a statement and, finally, his definitions of the quantifiers. We conclude in § 1.5.

1.2 The genesis of Popper's ideas on logic

In January 1937 Karl Popper arrived in New Zealand and settled down in Christchurch, where he had found employment as a lecturer of philosophy at Canterbury University College. It is in Christchurch where he worked on, and finished, what he considered to be his contribution to the war, "The Open Society and Its Enemies" (Popper, 1945). Combining teaching and research proved to be very difficult, and in his autobiography (Popper, 1974) he complains about how the leadership of the university actively discouraged research which was not directly related to his teaching activities. But, as he also writes in his autobiography, he found the time to work on logic (Popper, 1974, § 25). While he only started to publish his work once he had returned to Europe and worked at the London School of Economics, it is clear that most of the genesis of his novel ideas on logic can be traced back to his time in Christchurch.

Since the university library in Christchurch was poorly equipped, Popper also relied on the personal library of Henry George Forder, who taught mathematics at Auckland University College and who lent him journal articles and monographs that Popper needed for his logical and mathematical research. Popper started an extensive correspondence with Forder in 1943 which mostly turned around questions of the foundations of physics, mathematics and logic. The correspondence with his pre-war contacts from Europe, on the other hand, proved to be difficult and slow. One of his pre-war contacts that he did keep in contact with was Carnap, who taught at the University of Chicago and who sent Popper his latest publications in logic. Popper writes to Carnap to tell him that he received the "Introduction to Semantics" (Carnap, 1942) in October 1942 and "The Formalization of Logic" (Carnap, 1943) at the end of June or beginning of July 1943.

A significant part of his time was spent on preparing the courses that he taught, one of them being the introduction to formal logic. Popper was always keen on expressing his opinions as clearly as he could, and this attitude also applied to formal logic. We think that it is likely that the teaching of logic to his Christchurch students was the

occasion which prompted Popper to write down his thoughts on the foundations of logic. This is evidenced by the fact that he explicitly mentions discussions that he had with his student Peter Munz during one of his logic lectures in Christchurch (Popper, 1974, § 27 and endnote 194). After moving to England and taking up his new position at the London School of Economics he published the results obtained in New Zealand in a series of articles (Popper, 1947c,a,d, 1948b,c, 1949). At the same time he also thought about writing a textbook on logic that he could use in his lectures. He writes about this plan in a draft of a letter to Alexander Carr-Saunders, the director of the London School of Economics at the time:

> I may say that I am at present preparing a textbook on formal logic, not because I like writing a textbook (it interferes, on the contrary, badly with my own research programme) but because I find it necessary for my students. The existing textbooks have aims totally different from what I consider to be the aim of a modern introductory course in Logic. (Popper, 1946)

Indeed, already in 1939/41 Popper had prepared lecture notes on logic (Popper, 1941), and a table of contents for a textbook on logic can be found in Popper's estate (Popper, n.d.b). Moreover, together with Paul Bernays he wrote a manuscript "On Systems of Rules of Inference" (Popper and Bernays, n.d.) which contains an exposition of Popper's original approach to logic. The jointly written manuscript was not published, however.

1.3 Inferential definitions

In his approach to logic, Popper considers pairs of an *object language* $\mathcal{L}$ and a *deducibility relation* (also called *derivability relation*), written /, defined on $\mathcal{L}$. A given object language need not be a formal language but can also be a natural language. The deducibility relation between statements $a_1, \ldots, a_n$ and b is written as

$$a_1, \ldots, a_n/b$$

and is characterized by a so-called *basis*. Popper uses different bases. For clarity, we will use the following simple basis from Popper (1948b):

$$a_1, \ldots, a_n/a_i \quad (1 \leq i \leq n) \tag{Refl}$$
$$a_1, \ldots, a_n/b \rightarrow (b, a_1, \ldots, a_n/c \rightarrow a_1, \ldots, a_n/c) \tag{Trans}$$

The basis is formulated in a symbolic metalanguage, where $\rightarrow$ stands for "if-then". Further metalinguistic symbols are used, with the following meanings:

Symbol	$\rightarrow$	$\leftrightarrow$	&	(a)
Meaning	if-then	if and only if	and	for all a

Note that the axioms (Refl) and (Trans) are thus metalinguistic statements about the deducibility relation. They express that the deducibility relation / is reflexive and transitive. Besides these two structural properties nothing else characterizes the primitive notion of deducibility.

Popper distinguishes between a *general theory of derivation*, which deals with deducibility and related notions, and a *special theory of derivation*, in which logical constants are defined in terms of deducibility.

For example, in the general theory the relation of *mutual deducibility* // is defined in terms of deducibility / as follows:

$$a/\!/b \ \leftrightarrow \ (a/b \ \& \ b/a) \qquad\qquad \text{(mutual deducibility)}$$

This is an equivalence relation, and two mutually deducible statements a and b are said to have the same *logical force*. Thus, the equivalence classes induced by // are logical forces. Another important defined relation is *relative demonstrability*, written $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$:

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m \ \leftrightarrow \ (c)((b_1/c \ \& \ \ldots \ \& \ b_m/c) \rightarrow a_1, \ldots, a_n/c)$$
$$\text{(relative demonstrability)}$$

In words: The statements $b_1, \ldots, b_m$ are demonstrable relative to statements $a_1, \ldots, a_n$ (by definition) if, and only if, for all statements c: if c is deducible from each of the statements $b_1, \ldots, b_m$, then c is deducible from the statements $a_1, \ldots, a_n$ taken together. The notion of relative demonstrability is especially useful in cases where the object language contains conjunction $\wedge$ and disjunction $\vee$, since one can then show

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m \ \leftrightarrow \ a_1 \wedge \ldots \wedge a_n \vdash b_1 \vee \ldots \vee b_m$$

which gives us an interpretation of Gentzen's sequents (cf. Gentzen, 1935a). From this point of view, Popper's basis characterizes commas on the left side of $\vdash$ as conjunction and commas on the right side of $\vdash$ as disjunction. Furthermore, the notion of relative demonstrability contains as special cases the concepts of complementarity, demonstrability, contradictoriness and refutability, which Popper defines as well (cf. Binder and Piecha, 2017 for details).

The primitive notion of deducibility (and the notions defined in terms of it) is the foundation of Popper's *special theory of derivation*. In this theory, logical constants are defined in terms of deducibility alone. That is, a sign of a given object language is a logical constant, if, and only if, the sign can be defined by deducibility. Such definitions of logical constants (or *formative signs*, as Popper also calls them) are called *inferential definitions* by Popper:

> [. . .] inferential definitions [. . .] are characterized by the fact that they define a formative sign by its logical force which is defined, in turn, by a definition in terms of inference (i.e., of "/"). (Popper, 1947a, p. 286)

Inferential definitions of logical constants have the following form (where we use ∘ as a placeholder for an arbitrary binary connective):

$$a /\!/ a_1 \circ a_2 \;\leftrightarrow\; \mathcal{R}(a, a_1, a_2) \tag{D$\circ$}$$

In words: The object language statement a has the same logical force as the complex object language statement $a_1 \circ a_2$ if, and only if, the condition $\mathcal{R}(a, a_1, a_2)$ holds. Condition $\mathcal{R}(a, a_1, a_2)$ is a formula of the (symbolic) metalanguage containing (among others) the statements a, a_1, a_2 and the deducibility relation $/$ (or maybe relations like $\vdash$, which are defined in terms of $/$). Popper calls a definition of the form (D$\circ$) an *explicit definition* of the connective $\circ$. To simplify the presentation one can consider only the right part of such definitions, replacing a by $a_1 \circ a_2$ in $\mathcal{R}$:

$$\mathcal{R}(a_1 \circ a_2, a_1, a_2) \tag{C$\circ$}$$

This is called the *characterizing rule* (C$\circ$); it corresponds to the definition (D$\circ$).

As examples, we show some inferential definitions of connectives given by Popper:

Conjunction $\wedge$:

$$a /\!/ b \wedge c \;\leftrightarrow\; (d)(a \vdash d \;\leftrightarrow\; b, c \vdash d) \tag{D$\wedge$}$$
$$b \wedge c \vdash d \;\leftrightarrow\; b, c \vdash d \tag{C$\wedge$}$$

Disjunction $\vee$:

$$a /\!/ b \vee c \;\leftrightarrow\; (d)(d \vdash a \;\leftrightarrow\; d \vdash b, c) \tag{D$\vee$}$$
$$d \vdash b \vee c \;\leftrightarrow\; d \vdash b, c \tag{C$\vee$}$$

Conditional $>$:

$$a /\!/ b > c \;\leftrightarrow\; (d)(d \vdash a \;\leftrightarrow\; d, b \vdash c) \tag{D$>$}$$
$$d \vdash b > c \;\leftrightarrow\; d, b \vdash c \tag{C$>$}$$

Popper also considers several definitions for *classical negation* ($\neg_k$), among them the following two, which are equivalent:

$$a /\!/ \neg_k b \;\leftrightarrow\; (a, b \vdash \;\&\; \vdash a, b) \tag{D$\neg_k$1}$$
$$a /\!/ \neg_k b \;\leftrightarrow\; (c)(d)(d, a \vdash c \;\leftrightarrow\; d \vdash b, c) \tag{D$\neg_k$2}$$

The characterizing rules are the following:

$$\neg_k b, b \vdash \;\&\; \vdash \neg_k b, b \tag{C$\neg_k$1}$$
$$(c)(d)(d, \neg_k b \vdash c \;\leftrightarrow\; d \vdash b, c) \tag{C$\neg_k$2}$$

Other examples of unary connectives are the following:

Tautology t:

$$a /\!/ t(b) \;\leftrightarrow\; (c)(b/a \;\leftrightarrow\; c/a) \tag{Dt}$$
$$(c)(b/t(b) \;\leftrightarrow\; c/t(b)) \tag{Ct}$$

Contradiction f:

$$a /\!/ f(b) \leftrightarrow (c)(a/b \leftrightarrow a/c) \tag{D f}$$

$$(c)(f(b)/b \leftrightarrow f(b)/c) \tag{C f}$$

We have for all statements b: $\vdash t(b)$ and $f(b) \vdash$. In other words, t is a unary verum, and f is a unary falsum.

Popper's approach is not restricted to classical logic. For example, he inferentially defines several kinds of non-classical negations, such as

Intuitionistic negation $\neg_i$:

$$a /\!/ \neg_i b \leftrightarrow (c)(c \vdash a \leftrightarrow c, b \vdash) \tag{D$\neg_i$}$$

$$c \vdash \neg_i b \leftrightarrow c, b \vdash \tag{C$\neg_i$}$$

Popper does not only consider the availability of a characterizing rule like $\mathcal{R}(c, a_1, \ldots, a_n)$ as a criterion for the logicality of the constant characterized by it. Thus an inferential definition of this form need not define a logical constant in all cases. As a stronger criterion for logicality, Popper considers the existence of so-called *fully characterizing rules*, which are characterizing rules satisfying uniqueness in the sense that one can show that any two statements satisfying such a rule are mutually deducible (i.e., have the same logical force). In other words, a rule $\mathcal{R}(c, a_1, \ldots, a_n)$ is called *fully characterizing* if, and only if,

$$\mathcal{R}(a, a_1, \ldots, a_n) \,\&\, \mathcal{R}(b, a_1, \ldots, a_n) \rightarrow a /\!/ b.$$

The existence of fully characterizing rules is then used to distinguish between logical and non-logical constants (cf. the discussion in Schroeder-Heister, 1984, 2006 and Binder and Piecha, 2017, § 4.3).

1.4 Substitution, identity and quantification

We cannot say precisely when Popper's ideas about propositional logic took shape. In the introduction to "New Foundations for Logic" (Popper, 1947d) he writes that he obtained the results "during the last ten years", that is, between 1937 and 1947, roughly corresponding to the time he spent in New Zealand. On the other hand, we can give the exact date when he extended his inferential definitions to quantifiers. In a letter to Paul Bernays dated October 19th 1947 (Popper, 1947f) he writes:

> The first important result which I had finished about one week after I saw you, was the extension of the method of $a/b \wedge c \leftrightarrow a/b \,\&\, a/c$ to quantification.

The meeting that Popper refers to probably took place in Zürich on April 11th or 12th 1947[1], where Popper met Bernays in order to discuss the possibility of publishing a joint article on logic. The manuscript (Popper and Bernays, n.d.) for this unpublished article does not have a title; in a letter to Bernays dated March 3rd 1947 (Popper, 1947e), Popper suggests the title "On Systems of Rules of Inference", noting that "[t]he title is not very good, but so far I could not think of a better one".

Although they did not publish this manuscript, Popper's results found their way into several of his published articles. The most extensive discussion of these results can be found in §§ 7-8 of "New Foundations for Logic" (Popper, 1947d). Additionally, there is an important footnote in Popper (1948c), an alternative axiomatization in Popper (1947a), and a very short but clear summary of his treatment of quantification in Popper (1949). We follow the presentation of "New Foundations" (Popper, 1947d) but refer to some modifications which can be found in his other articles. Some modifications of his view on quantification were only discussed in hitherto unpublished correspondence[2], which we will discuss in § 1.4.4.

1.4.1 Formulas, name-variables and substitution

For propositional logic, as we saw in § 1.3, Popper considered pairs

$$(\mathcal{L}; a_1, \ldots, a_n / b)$$

of an object language $\mathcal{L}$ and a deducibility relation $/$, axiomatized by a basis consisting of the rules (Refl) and (Trans). Each element of the object language $\mathcal{L}$ was presumed to be a statement, that is, something which has a truth value.

The first modification Popper makes in order to treat quantification is to consider 4-tuples

$$(\mathcal{L}; \mathcal{P}; a_1, \ldots, a_n / b; a\binom{x}{y}))$$

consisting of a set $\mathcal{L}$ of *formulas*, a set $\mathcal{P}$ of *name-variables* (or pronouns), a *deducibility relation* on $\mathcal{L}$ and a *substitution operation*

$$a\binom{x}{y}$$

which substitutes the name-variable y for the name-variable x in the formula a. Variables $a, b, \ldots$ now range over *formulas* in $\mathcal{L}$, and variables $x, y, \ldots$ range over name-variables in $\mathcal{P}$.

Formulas can either be *open statements* (also called *statement-functions*) or *closed statements* (also called *statements*):

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[1] Bernays writes to Popper: "[...] nothing stands, as far as I can see, in the way of us seeing each other on April 11th in Zurich; I will certainly also be available in the midmorning of the 12th. I'm looking forward to the receipt of the concept you promised me, – also with regards to the possible joint publication." (Bernays, 1947)

[2] To be published in (Binder, Piecha, and Schroeder-Heister, 2021b).

formula

open statement,   closed statement,
statement-function   statement
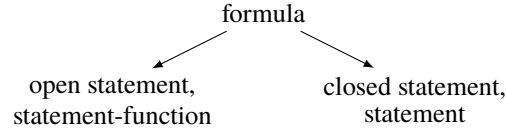
An example of an open statement given by Popper is "He is a charming fellow", which can be turned into a closed statement by replacing the name-variable "He" by the name "Ernest's best friend". Popper explicitly remarks that open statements do not have a truth value on their own; an open statement cannot be considered to be true or false.

The deducibility relation is axiomatized by the same rules (Refl) and (Trans) as in the case of propositional logic, but it now ranges over arbitrary formulas, not just closed statements. For example, Popper says that one can validly deduce the open statement "He is an excellent physician" from the open statement "He is not only a charming fellow but an excellent physician".

The new substitution operation is characterized by the four postulates (PF1) to (PF4) and the six primitive rules of derivation (6.1) to (6.6), which we present in a slightly simplified form in the following.

$$\mathcal{L} \cap \mathcal{P} = \emptyset \tag{PF1}$$

$$\text{If } a \in \mathcal{L} \text{ and } x, y \in \mathcal{P}, \text{ then } a\!\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \in \mathcal{L} \tag{PF2}$$

$$\text{For all } a \in \mathcal{L} \text{ there exists an } x \in \mathcal{P} \text{ such that for all } y \in \mathcal{P}: a\!\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)/\!/a \tag{PF3}$$

$$\text{There exist } a \in \mathcal{L} \text{ and } x, y \in \mathcal{P}: a/a\!\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \rightarrow t/f \tag{PF4}$$

Note that two kinds of metalinguistic quantifiers are used: There are universal and existential quantifiers ranging over statements $a \in \mathcal{L}$ and universal and existential quantifiers ranging over name-variables $x \in \mathcal{P}$. We only use symbols for the respective metalinguistic universal quantifiers in the following; (a) means "for all statements a" and (x) means "for all name-variables x".

The postulates (PF1) and (PF2) are, in a way, only about the correct grammatical use of formulas and name-variables. The postulate (PF3) says that for every formula there is some name-variable not occurring in it. This is obvious if the set of name-variables is considered to be infinite, and if each formula is a finite object which can only mention a finite number of name-variables. The postulate (PF4), which Popper considers to be optional, excludes degenerate systems in which only one object exists. Take, for example, the open statement a to be "x likes the current weather". The deducibility of "y likes the current weather" from "x likes the current weather" only leads to a contradiction if there are at least two persons to whom x and y can refer. Postulate (PF4) was also discussed in correspondence between Popper and Carnap (Carnap, 1947, Popper, 1947b; cf. Appendix A and Appendix B).

The six primitive rules of inference are given below. We will not discuss them in detail, but the reader may check that they are valid for a concrete formalized object language and a substitution operation for that language.

$$\text{If, for every } z, \ a/\!/a\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right) \text{ and } b/\!/b\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right), \text{ then } a/\!/b \rightarrow a\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)/\!/b\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \tag{6.1}$$

$$a/\!/a\left(\begin{smallmatrix}x\\x\end{smallmatrix}\right) \tag{6.2}$$

$$\text{If } x \neq y, \text{ then } \left(a\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)\left(\begin{smallmatrix}x\\z\end{smallmatrix}\right)/\!/a\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \tag{6.3}$$

$$\left(a\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)/\!/\left(a\left(\begin{smallmatrix}x\\z\end{smallmatrix}\right)\right)\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right) \tag{6.4}$$

$$\left(a\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)\left(\begin{smallmatrix}z\\y\end{smallmatrix}\right)/\!/\left(a\left(\begin{smallmatrix}z\\y\end{smallmatrix}\right)\right)\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \tag{6.5}$$

$$\text{If } w \neq x, \ x \neq u \text{ and } u \neq y, \text{ then } \left(a\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)\left(\begin{smallmatrix}u\\w\end{smallmatrix}\right)/\!/\left(a\left(\begin{smallmatrix}u\\w\end{smallmatrix}\right)\right)\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \tag{6.6}$$

The rules (6.1) to (6.6) characterize substitution as a structural operation; this is similar to how the basis characterizes commas in sequences of statements. It is remarkable that Popper here presents an algebraic treatment of substitution, which can be compared to the theory of explicit substitution developed much later (cf., e.g., Abadi et al., 1991).

As an intriguing sidenote, Popper compares the definition of substitution by the rules (6.1) to (6.6) to the definition of conjunction via the inferential definition (D∧). He writes:

> These six primitive rules determine the meaning of the symbol "$a\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$" in a way precisely analogous to the way in which, say, [rule (D∧) determines] the meaning of conjunction [. . .] with the help of the concept of derivability "/". (Popper, 1947d, p. 226)

However, it has been pointed out by Schroeder-Heister (1984, p. 106) that Popper's rules for substitution "cannot be brought into the form of an inferential definition of an operator of the object language". Hence, substitution cannot have the status of a logical constant according to Popper's criterion for logicality; his rules for substitution do not have the form of characterizing rules (and, consequently, no fully characterizing rules can be given either). Indeed, Popper also explains substitution as follows:

> The notation
>
> $$\text{"}a\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\text{"}$$
>
> will be used as a (variable) metalinguistic name of the statement which is the result of substituting, in the statement a (open or closed), the variable y for the variable x, wherever it occurs. $a\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$ is identical with a if x does not occur in a. (Popper, 1947a, p. 1216)

Popper's rules for substitution may thus be viewed as "an implicit characterization of a metalinguistic operation" (Schroeder-Heister, 1984, p. 106), and not as an inferential definition of a logical constant for object languages.

Next we discuss some auxiliary concepts defined with the help of both the deducibility relation and the substitution operation.

1.4.2 Non-dependence, identity and difference

If we work with some inductively defined formal object language, then we can easily specify the set of free variables of a formula by recursion on the structure of that formula. This possibility is excluded in Popper's approach, since its generality does not restrict us to the consideration of formal languages. Popper therefore defines the concept

$$a_{\dot{x}}$$

which can be read as "x does not occur among the free variables in a". Popper himself expresses this as "a does not depend on x", "a-*without*-x" and "x does not occur relevantly in a". The formula a does not depend on x if, and only if, substitution of some name-variable y for x does not change the logical strength of a. That is:

$$a // a_{\dot{x}} \leftrightarrow \text{ for every } y: a // a\binom{x}{y} \qquad (\text{D } a_{\dot{x}})$$

The second concept Popper defines with the help of deducibility and substitution is *identity*. As Popper (1947d, p. 227f, fn 24) notes, one first has to extend the object language $\mathcal{L}$ to incorporate formulas of the form $Idt(x, y)$; this is achieved by the postulate

$$\text{If } x \text{ and } y \text{ are name variables, then } Idt(x, y) \text{ is a formula} \qquad (\text{P } Idt)$$

In addition, the characterizing rules for substitution have to be extended by rules of the form

$$(Idt(x, y))\binom{x}{z} // Idt(z, y) \qquad (\text{A})$$

$$(Idt(x, y))\binom{y}{z} // Idt(x, z) \qquad (\text{B})$$

$$\text{If } x \neq u \neq y, \text{ then } Idt(x, y)\binom{u}{z} // Idt(x, y) \qquad (\text{C})$$

With these preliminaries, Popper defines identity using the following idea:

> The identity statement "$Idt(x, y)$" can be defined as the weakest statement strong enough to satisfy the [. . .] formula [. . .]
>
> $$\text{"}Idt(x, y), a(x)/a(y)\text{"}$$
>
> that is to say, the formula corresponding to what Hilbert-Bernays call the second identity axiom. (Hilbert-Bernays's first axiom follows from the demand that the identity statement must be the *weakest* statement satisfying this formula.) (Popper, 1949, p. 725f)

The identity axioms Popper refers to are the axioms J_1 and J_2 of Hilbert and Bernays (1934, p. 164):

$$a = a \qquad (J_1)$$

$$a = b \rightarrow (A(a) \rightarrow A(b)) \qquad (J_2)$$

This justifies the following definition of *identity Idt*(x, y):

$$a/\!/Idt(x, y) \; \leftrightarrow \; (\text{for every } b \text{ and } z: ((b/\!/b_{\dot{x}} \;\&\; b/\!/b_{\dot{y}}) \rightarrow a, b\binom{z}{x}/b\binom{z}{y})) \;\&$$
$$((\text{for every } c \text{ and } u: ((c/\!/c_{\dot{x}} \;\&\; c/\!/c_{\dot{y}}) \rightarrow b, c\binom{u}{x}/c\binom{u}{y}))) \rightarrow b/a)) \quad (\text{D}\,Idt)$$

Popper (1948c, p. 323f, fn 11) expands on the definition of identity *Idt*(x, y) in order to illustrate his method of obtaining a relatively simple characterizing rule from an explicit definition that is the weakest (or strongest) statement satisfying a certain condition or axiom. He first introduces the following abbreviating notation:

$$a/\!/a_{\dot{x}\dot{y}} \; \leftrightarrow \; (w)(a/\!/a\binom{x}{w} \;\&\; a/\!/a\binom{y}{w}).$$

Using this abbreviation, he defines *Idt*(x, y) as the weakest statement strong enough to imply the axiom J_2:

$$a/\!/Idt(x, y) \; \leftrightarrow$$
$$(b)(z)((b/\!/b_{\dot{x}\dot{y}} \rightarrow a, b\binom{z}{x}/b\binom{z}{y}) \;\&\; (((c)(u)(c/\!/c_{\dot{x}\dot{y}} \rightarrow b, c\binom{u}{x}/c\binom{u}{y}))) \rightarrow b/a))$$
$$(\text{D}\,Idt^{\dagger})$$

This explicit definition, which is an abbreviated version of (D *Idt*), can be replaced by a definition that corresponds to the following characterizing rule:

$$a/Idt(x, y) \; \leftrightarrow \; (b)(z)(b/\!/b_{\dot{x}\dot{y}} \rightarrow a, b\binom{z}{x}/b\binom{z}{y}) \qquad (\text{C}\,Idt^{\ddagger})$$

This can be seen by instantiating a in (D *Idt*†) with *Idt*(x, y) in order to obtain

$$(b)(z)((b/\!/b_{\dot{x}\dot{y}} \rightarrow Idt(x, y), b\binom{z}{x}/b\binom{z}{y}) \;\&$$
$$(((c)(u)(c/\!/c_{\dot{x}\dot{y}} \rightarrow b, c\binom{u}{x}/c\binom{u}{y}))) \rightarrow b/Idt(x, y))).$$

The left conjunct gives the direction from left to right in (C *Idt*‡), and the right conjunct gives the direction from right to left.

Finally, *difference Dff*(x, y) is simply defined as the classical negation of identity:

$$a/\!/Dff(x, y) \; \leftrightarrow \; a/\!/\neg_k Idt(x, y) \qquad (\text{D}\,Dff)$$

It is interesting to see that Popper chose to treat occurrence of free variables and identity as defined notions, rather than to class them with substitution and deducibility among the primitive notions characterized by the basis. We will see further on (cf. § 1.4.4) that Popper probably revised this position later.

1.4.3 Quantification

Inferential definitions of universal and existential quantification are introduced in (Popper, 1947d), to which he later published a list of corrections and additions (Popper, 1948a), which we take into account here. Popper's aim is not to develop and analyze the theory of quantification, that is, first-order logic, but to show that his approach to quantification is at least on a par with other proposed treatments of quantification. He therefore restricts himself to stating his definitions of the quantifiers and to deriving some simple conclusions, but he does not formally develop a meta-theory of quantification. He does not, for example, discuss the completeness of his rules, the difference between classical and constructive interpretations of the existential quantifier, or the relation to models of his system.

Later, Popper (1949) gives the clearest explanation of what intuition his inferential definition of *universal quantification* is supposed to capture. He writes:

> The result of universal quantification of a statement *a* can be defined as the weakest statement strong enough to satisfy the law of specification, that is to say, the law "what is valid for all instances is valid for every single one". (Popper, 1949, p. 725)

Presupposing his rules of substitution, and writing Ax for the universal quantifier, Popper's inferential definition and the characterizing rule for universal quantification are the following:

$$a_{\grave{y}} /\!/ Axb_{\grave{y}} \;\leftrightarrow\; (\text{for every } c_{\grave{y}}\colon c_{\grave{y}}/a_{\grave{y}} \;\leftrightarrow\; c_{\grave{y}}/b_{\grave{y}}\tbinom{x}{y})) \qquad\qquad \text{(D7.1)}$$

$$\text{For every } c_{\grave{y}}\colon c_{\grave{y}}/Axb_{\grave{y}} \;\leftrightarrow\; c_{\grave{y}}/b_{\grave{y}}\tbinom{x}{y} \qquad\qquad \text{(C7.1)}$$

In order to see how more ordinary presentations of the rules for universal quantification follow from these inferential definitions, we can compare them to the more familiar rules of the (intuitionistic) sequent calculus (writing $\varphi[x/y]$ for the result of substituting y for x in the formula φ):

$$\frac{\Gamma, \varphi[x/t] \vdash \psi}{\Gamma, \forall x\varphi \vdash \psi} \;(\forall \vdash) \qquad\qquad\qquad \frac{\Gamma \vdash \varphi[x/y]}{\Gamma \vdash \forall x\varphi} \;(\vdash \forall)$$

$$\frac{\Gamma, \varphi[x/y] \vdash \psi}{\Gamma, \exists x\varphi \vdash \psi} \;(\exists \vdash) \qquad\qquad\qquad \frac{\Gamma \vdash \varphi[x/t]}{\Gamma \vdash \exists x\varphi} \;(\vdash \exists)$$

with the variable condition that y does not occur free in the conclusion of $(\vdash \forall)$ and $(\exists \vdash)$.

For example, by instantiating (C7.1) with $Axb_{\grave{y}}$ and by using the rules (Trans) and (Refl) from the basis, we obtain the following rule

$$a, b_{\grave{y}}\tbinom{x}{y}/c \;\rightarrow\; a, Axb_{\grave{y}}/c$$

which can easily be seen to be a variant of the rule $(\forall \vdash)$ where the name-variable y takes the role of the term t. Similarly, by instantiating (C7.1) with $c_{\grave{y}}$ and reading the

biimplication from right to left we obtain the following rule, which corresponds to the rule ($\vdash \forall$) with the variable condition that y does not occur relevantly in c:

$$c_{\bar{y}}/b_{\bar{y}}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \rightarrow c_{\bar{y}}/Axb_{\bar{y}}.$$

As was the case for universal quantification, Popper gives the clearest explanation of the inferential definition of *existential quantification* not in (Popper, 1947d), but in (Popper, 1949, p. 725):

> The result of existential quantification of the statement a can be defined as the strongest statement weak enough to follow from every instance of a.

The inferential definition and the characterizing rule for the *existential quantifier Ex* are

$$a_{\bar{y}}/\!/Exb_{\bar{y}} \;\leftrightarrow\; (\text{for every } c_{\bar{y}}\colon a_{\bar{y}}/c_{\bar{y}} \;\leftrightarrow\; b_{\bar{y}}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)/c_{\bar{y}}) \tag{D7.2}$$

$$\text{For every } c_{\bar{y}}\colon Exb_{\bar{y}}/c_{\bar{y}} \;\leftrightarrow\; b_{\bar{y}}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)/c_{\bar{y}} \tag{C7.2}$$

To elucidate, we derive some more familiar rules for the existential quantifier from its characterizing rule. Instantiating (C7.2) with $Exb_{\bar{y}}$ and using the rules of the basis we can obtain the rule

$$a/b_{\bar{y}}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \rightarrow a/Exb_{\bar{y}}$$

which corresponds to the sequent calculus rule ($\vdash \exists$); and by instantiating (C7.2) with $c_{\bar{y}}$ and reading the biimplication from right to left, we obtain the following rule, which corresponds to ($\exists \vdash$):

$$b_{\bar{y}}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)/c_{\bar{y}} \rightarrow Ex_{\bar{y}}/c_{\bar{y}}.$$

Popper does not consider the explicit definitions (D7.1) and (D7.2) to be improvements compared to the characterizing rules. They are given to show that universal and existential quantification can be defined using only his basis and the rules (6.1) to (6.6). He notices that these rules are not as simple as the rules of his basis, for example. But he points out that the concept of "$a_{\bar{x}}$" can be avoided in these definitions (Popper, 1947d, p. 230, fn 26, added in the corrections and additions Popper, 1948a). Assuming $x \neq y$, one can use instead:

$$a\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right)/Ax(b\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right)) \;\leftrightarrow\; a\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right)/b\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \tag{7.1*}$$

$$Ex(a\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right))/b\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right) \;\leftrightarrow\; a\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)/b\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right) \tag{7.2*}$$

$$a\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right)/\!/Ax(b\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right)) \;\leftrightarrow\; (\text{for every } c\colon c\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right)/a\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right) \;\leftrightarrow\; c\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right)/b\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)) \tag{D7.1*}$$

$$a\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right)/\!/Ex(b\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right)) \;\leftrightarrow\; (\text{for every } c\colon a\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right)/c\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right) \;\leftrightarrow\; b\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)/c\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right)) \tag{D7.2*}$$

He conceives his rules of quantification to be less complicated than those given by Hilbert and Ackermann (1928) or those given by Quine (1940, § 15), and he emphasizes that his rules in the end make use of only one logical concept, namely that of deducibility / as characterized by his basis.

1.4.4 An unfortunate misunderstanding

Popper (1947d, § 8) introduces a distinction which he considered to be very important: the distinction between rules of derivation and rules of proof. If he had not stopped publishing in logic, then it is very likely that he would have developed these ideas in more detail. For example, among his unpublished manuscripts there are two which are entitled "Derivation and Demonstration in Propositional and Functional Logic" (Popper, n.d.a) and "The Propositional and Functional Logic of Derivation and of Demonstration" (Popper, n.d.c), as well as another untitled manuscript (Popper, n.d.d), which also deals with the distinction between derivation and demonstration.

In order to illustrate this distinction we have to make use of the concept of relative demonstrability $\vdash$, which was introduced in § 1.3. If we specialize this concept to no formula on the left hand side and exactly one formula on the right hand side, we obtain the definition of a *provable formula a*: $\vdash a$. Consider now the following two formulas of the metalanguage:

$$a/b \rightarrow (\vdash a \rightarrow \ \vdash b)$$
$$(\vdash a \rightarrow \ \vdash b) \rightarrow a/b.$$

Popper correctly remarks that while the first formula is valid, the second is not. This can be seen by instantiating a by a consistent formula and b by a contradictory one. Now Popper correctly observes that the rules of a system like Principia Mathematica (Whitehead and Russell, 1927) are rules of proof and not rules of derivation. For example, the rule of modus ponens takes the form

$$\vdash a \rightarrow (\vdash a > b \rightarrow \ \vdash b)$$

rather than the form

$$a, a > b/b.$$

What Popper intends to formulate here, and in particular in his definition of a purely derivational system of primitive rules (cf. Popper, 1947d, definition (D8.1)), is, in our opinion, a criterion that allows to distinguish between formulations of logic based on axioms and rules of proof, such as Hilbert and Bernays's (1934; 1939) or Whitehead and Russell's (1927) on the one hand, and formulations of logic based on derivation alone, such as Gentzen's (1935a; 1935b) and his own, on the other hand.

Unfortunately, he applied this analysis of rules of derivation and rules of proof to the systems of Carnap as well as of Hilbert and Bernays in a way that does not take account of an important difference between his system and theirs. Popper (1947d, p. 232) warns that there are rules of proof such as

$$\vdash a_{\hat{y}} \ \leftrightarrow \ \vdash a_{\hat{y}} \begin{pmatrix} x \\ y \end{pmatrix} \tag{8.5}$$

which are valid, whereas the corresponding rule of derivation

$$a_{\hat{y}}/a_{\hat{y}} \begin{pmatrix} x \\ y \end{pmatrix}$$

is invalid. He continues:

> Now all the mistakes here warned against do actually vitiate some otherwise
> very excellent books on modern logic – an indication that the distinction
> between (conditional) rules of proof or rules of demonstration on the one side
> and rules of derivation on the other cannot be neglected without involving
> oneself in contradictions. (Popper, 1947d, p. 233)

Both Carnap and Bernays responded to Popper's criticism of their respective system
in correspondence. We have reproduced Carnap's letter and Popper's response in
Appendix A and Appendix B, respectively. Bernays (1948) writes:

> Now I have to comment upon your critique of the formulation of the all-
> schema, as it is given in the "Grundlagen der Math." [Hilbert and Bernays
> (1934)]. I think of the passage p. 232–233 of your New Foundations. [. . .] The
> contradiction that you derive, starting with the schema $a_{\tilde{x}} > b/a_{\tilde{x}} > Axb$
> which you criticize, does not arise in the formalism of the "Grundl. der Math.",
> because the implication plays another role here than the "hypothetical" in
> your formalism.

We note that Popper's letter to Carnap (cf. Appendix B) is also interesting for the
fact that it contains an expansion of his theory of quantification by presenting several
logical laws of classical first-order logic.

Popper later revised his understanding of the interaction of substitution and
deducibility. While his definitions are formulated using the weaker notion of interde-
ducibility, he then considered it necessary to use the stronger notion of identity of
statements (Popper, 1974, p. 171, endnote 198; reproduced in Appendix C).

Concerning possible future work on logic, Popper states in his reply (Popper,
1948d) to Bernays's letter (Bernays, 1948):

> I have also a number of new results – but I do not believe that I will ever
> dare again to publish something (except, maybe, an infinite sequence of
> corrections to my old publications)!

1.5 Conclusion

Popper's works on logic in the 1940s had no real influence on the further development
of logic. This is despite the fact that he anticipated and had results on several issues
in the area of philosophical logic which are still discussed today. We mention his
inferentialist approach to logic, his analysis of logicality, and his results on combining
logical systems (cf. Binder and Piecha, 2017; Schroeder-Heister, 1984, 2006 for
details). In his inferentialist approach to logic, Popper anticipated many ideas of
proof-theoretic semantics (Schroeder-Heister, 2018; Piecha and Schroeder-Heister,
2016; cf. Binder, Piecha, and Schroeder-Heister, 2021a).

At the time, his works were reviewed by several prominent logicians, including
Ackermann (1948, 1949a,b), Beth (1948), Curry (1948a,b,c,d, 1949), Hasenjaeger

(1949), Kleene (1948, 1949) and McKinsey (1948). While the reviews by Ackermann and Beth are summarizing, Curry, Hasenjaeger, Kleene and McKinsey are critical about certain aspects of Popper's approach and point out some technical issues (for a discussion of these criticisms cf. Schroeder-Heister, 1984). Concerning Popper's treatment of quantification in particular, Curry (1948a) raised some doubts (which were also discussed by Seldin, 2008), although without going into details and while maintaining that "[p]resumably [Popper's] ideas can be carried through, at least in principle". Popper also disseminated his ideas in correspondence with Carnap, for example, who saw the importance of Popper's work on logic (Carnap, 1947; cf. Appendix A). Brouwer (1947) reacted positively as well and presented Popper's articles (1947a; 1948b; 1948c) to the Koninklijke Nederlandse Akademie van Wetenschappen for publication. However, although Popper's approach to logic is original and philosophically quite interesting, it did not receive the wider appreciation it deserves.

Appendix A Letter from Carnap to Popper, October 9th 1947 (Carnap, 1947)

<div align="right">

Santa Fe, N.M.,
P.O.B. 1214
Oct. 9, 1947

</div>

Dear Popper,

My best thanks for your letter of August & your kind judgement on my book.

I just read your "New Found." in "Mind". It is very interesting & contains a number of new & important results. It is an essential improvement in comparison with my previous attempts of defining the connection in terms of "consequence" (first in "Syntax" § 57, & later in "Formalization"). Among other things, your clear & simple analysis of the three kinds of negation is very valuable.

Your discussion on pp. 232f. is, unfortunately, so short that I am unable to understand it. I should like to understand it, especially since it is the basis of your objections against my rules. You say that the last formula on p. 232 leads to that on top of p. 233. How does it? Is the restriction $a_{\bar{x}}$ in the former but not in the latter no impediment? Further, you say that $a//a\binom{x}{y}$ violates PF2. How does it? (I say that it violates your interpretation.) This is only a question, not an objection; I assume that your assertions concerning *your* system are correct.

However, I doubt very much whether your assertions (in the footnotes p. 232) of the non-validity of my rules 10 & 11 in D28-2 are correct. Note that rule 11, because of its restriction, does not lead to a proof of "$Px \supset (x)(Px)$" (which would indeed be wrong), although "$(x)(Px)$" is derivable from "Px", in distinction to your system. You have probably made the mistake of inadvertently transforming the interpretation & the rules of your system to mine (+ Hilbert's, etc.) Perhaps you have overlooked the following essential distinction. In my system (& Hilbert's, etc. but perhaps not in Princ. Math., & certainly not in your system), "Px" (as a separate formula) is interpreted (see, e.g., "Syntax", p. 22, par. 2) as meaning the same as "Py" & as "$(x)(Px)$". Therefore my rules are valid, if you doubt it, please give a counter-example, by using only *my* rules, not yours.

Feigl & Hempel (& his new wife) were here for a few weeks, & we had a very nice time together, with many interesting discussions, mostly on inductive logic.

We shall stay here until Xmas.

> With best regards,
> yours,
> Rudolf Carnap

(Please, let's forget about titles.)

Appendix B Letter from Popper to Carnap, November 24th 1947 (Popper, 1947b)

> November 24th, 1947.

Dear Carnap,

I am sending you to-day an offprint of my "Logic without Assumptions", referred to in my "New Foundations" (note 1 on p. 203). Two more papers are on the way; I have been promised the offprints of one of them for next week, and I shall send you a copy at once.

I am overworked (8 hours lecturing a week is too much if one does research – I wish I could get some time off for research, but I don't know how), and really quite exhausted.

You asked me in your last letter for a fuller explanation of my pp. 232f. (of my "New Foundations"). I suppose that it is the misprint on p. 233 ("PF2" should properly read "PF4") which created the difficulty, and that you will have found meanwhile what I meant. Still, here is a fuller explanation.

My contention is this.

Your statement (*Formalization*, p. 136) "that there exists a one-one correlation between the individuals and the natural numbers" indicates that it is your intention to construct a calculus which is *consistent with* my (much weaker) postulate PF4, i.e., with the demand that there exists *more than one individual*.

But with the assumption that there exists more than one individual, each of the following rules of your *Formalization* contradict:

C10 (i.e., D28-2, rule 10, on p. 137)
C11 (i.e., D28-2, rule 11, on p. 138)
C12 (i.e., D28-2, rule 12, on p. 138)
Cb   (i.e., T28-4, case b,   on p. 139).

For the proof of this contention, I shall make use of my own formalism. But the proof holds for your formalism as well; for your C-implication satisfies, on the basis of your *Introduction*, p. 64, P14-5; P14-8; and P14-11, all the rules which define my "... / ...", i.e., the rules which I shall call "generalized reflexivity principle" and "generalized transitivity principle". To the latter, I shall refer as "Tg".

I shall also refer to the following principles ("a^k" is the classical negation of a):

(1.1) $a/b \rightarrow \ \vdash a > b$

(1.2) $a/b \rightarrow (\vdash a \rightarrow \ \vdash b)$

(1.3) $\vdash a > b \ \leftrightarrow \ \vdash b^k > a^k$

(1.4) $(a \wedge a^k) \vee b // b$

(1.5) $a // a^{kk}$

(1.6) $(Ax(a^k))^k // Exa$ (cp. your d and e, p. 139)

I begin with C10, which I write

(C10) $a/a\binom{x}{y}$, provided y is not bound in $a\binom{x}{y}$.

We obtain, always assuming that y is not bound in $a\binom{x}{y}$:

(C10.1) $\vdash a > a\binom{x}{y}$ (C10;1.1)

(C10.2) $\vdash (a\binom{x}{y})^k > a^k$ (C10;1;1.3)

(C10.3) $(a\binom{x}{y})^k / a^k$ (C10.2;1.1)

(C10.4) $\vdash (a\binom{x}{y})^k \rightarrow \ \vdash a^k$ (C10.3;1.2)

Now we take "a" to be the name of an open statement (such as "$x + 1 = y$") which is *satisfiable but not* universally true. We obtain

(C10.5) $\vdash$ "$y + 1 \neq y$" $\rightarrow \ \vdash$ "$x + 1 \neq y$"   (C10.4)

and, substituting further "$x + 1$" for "y" (we *may* confine this to the right hand side, but I shall do it throughout) we obtain

(C10.6) $\vdash$ "$(x + 1) + 1 \neq x + 1$" $\rightarrow \ \vdash$ "$x + 1 \neq x + 1$"   (C10.5)

If there exists only *one* individual, then every statement of the form "... $\neq$ —" is false, and C10.6 is innocuous. But if there are more individuals than one, C10.6 gives rise to

(C10.7)   "$x + 1 \neq x + 1$"

which is contradictory.

————

I now proceed to C11. This may be written:

(C11) $a \vee b / a \vee Axb$, provided x is not free in a.

We obtain, substituting "$Axc \wedge (Axc)^k$" for "a":

(C11.1)	$(Axc \wedge (Axc)^k) \vee b/(Axc \wedge (Axc)^k) \vee Axb$	(C11)
(C11.2)	b/Axb	(C11.1;1.4;Tg.)
(C11.3)	$\vdash b > Axb$	(C11.2;1.1)
(C11.4)	$\vdash (Axb)^k > b^k$	(C11.3;1.3)
(C11.5)	$\vdash (Ax(a^k))^k > (a^k)^k$	(C11.4)
(C11.6)	$(Ax(a^k))^k/a^{kk}$	(C11.5;1.1)
(C11.7)	Exa/a	(C11.6;1.5;1.6;Tg.)
(C11.8)	Exa/Axa	(C11.7;C11.2;Tg.)

But C11.8 is, clearly, satisfied only if there is not more than *one* individual.

————

I proceed to rule C12. This may be written

(C12) $a^k \vee b/(Exa)^k \vee b$, provided x is not free in b.

Substituting "$Axc \wedge (Axc)^k$" for "b" (as before under C11), we obtain:

(C12.1)	$a^k/(Exa)^k$	(C12)
(C12.2)	$b^{kk}/(Ex(b^k))^k$	(C12.1)
(C12.3)	b/Axb	(C12.2;1.5;1.6;Tg.)

But C12.3 is the same as C11.2, and has the same fatal consequences.

————

Rule Cb, of course, is also the same as C11.2 and C12.3.
The result of all this is:
(1) Rule Cb can be dropped altogether.
(2) The rules of derivation C10; C11; and C12 must be replaced by the corresponding conditional rules of proof, C′10'; C′11; C′12:

(C′10) $\vdash a \rightarrow \vdash a\binom{x}{y}$, provided y is not bound in $a\binom{x}{y}$.

(C′11) $\vdash a \lor b \rightarrow\ \vdash a \lor Axb$, provided x is not free in a.

(C′12) $\vdash a^k \lor b \rightarrow\ \vdash (Exa)^k \lor b$, provided x is not free in b.

The last two rules may be replaced by C″11 and C″12:

(C″11) $a\binom{x}{y}/b \rightarrow a\binom{x}{y}/Axb$, provided $x \neq y$

(C″12) $a/b\binom{x}{y} \rightarrow Exa/b\binom{x}{y}$, provided $x \neq y$.

These two rules, in turn, can be replaced by:

(C‴11) $a\binom{y}{x}/b\binom{x}{y} \rightarrow a\binom{y}{x}/Axb\binom{y}{x}$ $(x \neq y)$

(C‴12) $a\binom{x}{y}/b\binom{y}{x} \rightarrow Exa\binom{y}{x}/b\binom{y}{x}$ $(x \neq y)$

If we replace here "$\rightarrow$" by "$\leftrightarrow$", we obtain the rules which define the quantifiers, and from which, in the presence of the six rules defining "$a\binom{x}{y}$", everything else can be obtained:

(C⁗) $a\binom{y}{x}/Axb\binom{y}{x} \leftrightarrow a\binom{y}{x}/b\binom{x}{y}$ $(x \neq y)$

(C⁗) $Exa\binom{y}{x}/b\binom{y}{x} \leftrightarrow a\binom{x}{y}/b\binom{y}{x}$ $(x \neq y)$

Appendix C Popper (1974, p. 171, endnote 198)

The mistake was connected with the rules of substitution or replacement of expressions: I had mistakenly thought that it was sufficient to formulate these rules in terms of interdeducibility, while in fact what was needed was identity (of expressions). To explain this remark: I postulated, for example, that if in a statement a, two (disjoint) subexpressions x and y are both, wherever they occur, replaced by an expression z, then the resulting expression (provided it is a statement) is interdeducible with the result of replacing first x wherever it occurs by y and then y wherever it occurs by z. What I should have postulated was that the first result is identical with the second result. I realized that this was stronger, but I mistakenly thought that the weaker rule would suffice. The interesting (and so far unpublished) conclusion to which I was led later by repairing this mistake was that there was an essential difference between propositional and functional logic: while propositional logic can be constructed as a theory of sets of statements, whose elements are partially ordered by the relation of deducibility, functional logic needs in addition a specifically morphological approach since it must refer to the subexpression of an expression, using a concept like identity (with respect to expressions). But no more is needed than the ideas of identity and subexpression; no further description especially of the shape of the expressions.

References

Abadi, M., Cardelli, L., Curien, P.L., Levy, J.J.: Explicit Substitutions. Journal of Functional Programming **1**, 375–416 (1991)

Ackermann, W.: Review of Functional Logic without Axioms or Primitive Rules of Inference. Zentralblatt für Mathematik und ihre Grenzgebiete **29**, 196 (1948)

Ackermann, W.: Review of On the Theory of Deduction, Part I. Zentralblatt für Mathematik und ihre Grenzgebiete **30**, 3 (1949a)

Ackermann, W.: Review of On the Theory of Deduction, Part II. Zentralblatt für Mathematik und ihre Grenzgebiete **30**, 101 (1949b)

Bernays, P.: Unpublished letter to K. R. Popper, March 12th 1947 (1947). Karl Popper-Sammlung, Universität Klagenfurt, Box 276, Folder 12. Translation by the authors.

Bernays, P.: Unpublished letter to K. R. Popper, May 12th 1948 (1948). ETH Zürich, Paul Bernays Collection, Hs 975 : 3653. Translation by the authors.

Beth, E.W.: Review of New Foundations for Logic. Mathematical Reviews **9**, 130 (1948). URL http://www.ams.org/mathscinet-getitem?mr=0021924

Binder, D., Piecha, T.: Popper's Notion of Duality and His Theory of Negations. History and Philosophy of Logic **38**(2), 154–189 (2017)

Binder, D., Piecha, T., Schroeder-Heister, P.: Popper's Theory of Deductive Logic (2021a). Introduction to Popper's articles on logic in: Binder, Piecha, and Schroeder-Heister (2021b)

Binder, D., Piecha, T., Schroeder-Heister, P.: The Logical Works of Karl Popper. Springer (2021b)

Brouwer, L.E.J.: Letter to K. R. Popper. December 10th, 1947, handwritten (1947). Karl Popper-Sammlung, Universität Klagenfurt, Box 280, Folder 8.

Carnap, R.: Introduction to Semantics. Harvard University Press (1942)

Carnap, R.: Formalization of Logic. Harvard University Press, Cambridge, Mass. (1943)

Carnap, R.: Letter to K. R. Popper, October 9th 1947 (1947). Karl Popper-Sammlung, Universität Klagenfurt, Box 282, Folder 24.

Curry, H.B.: Review of Functional Logic without Axioms or Primitive Rules of Inference. Mathematical Reviews **9**, 321 (1948a). URL http://www.ams.org/mathscinet-getitem?mr=0023207

Curry, H.B.: Review of Logic without Assumptions. Mathematical Reviews **9**, 486 (1948b). URL http://www.ams.org/mathscinet-getitem?mr=0024397

Curry, H.B.: Review of On the Theory of Deduction, Part I. Mathematical Reviews **9**, 486–487 (1948c). URL http://www.ams.org/mathscinet-getitem?mr=0024398

Curry, H.B.: Review of On the Theory of Deduction, Part II. Mathematical Reviews **9**, 487 (1948d). URL http://www.ams.org/mathscinet-getitem?mr=0024399

Curry, H.B.: Review of The Trivialization of Mathematical Logic. Mathematical Reviews **10**, 422 (1949). URL http://www.ams.org/mathscinet-getitem?mr=0028263

Gentzen, G.: Untersuchungen über das logische Schließen, I. Mathematische Zeitschrift **39**(1), 176–210 (1935a). DOI 10.1007/BF01201353. URL http://dx.doi.org/10.1007/BF01201353. English Translation in Gentzen (1969).

Gentzen, G.: Untersuchungen über das logische Schließen, II. Mathematische Zeitschrift **39**(1), 405–431 (1935b). DOI 10.1007/BF01201363. URL http://dx.doi.org/10.1007/BF01201363. English Translation in Gentzen (1969).

Gentzen, G.: Collected Papers of Gerhard Gentzen. North-Holland, Amsterdam (1969). Edited by M. E. Szabo.

Hasenjaeger, G.: Review of The Trivialization of Mathematical Logic. Zentralblatt für Mathematik und ihre Grenzgebiete **31**, 193 (1949)

Hilbert, D., Ackermann, W.: Grundzüge der theoretischen Logik. Springer (1928)

Hilbert, D., Bernays, P.: Grundlagen der Mathematik, vol. 1. Springer (1934)

Hilbert, D., Bernays, P.: Grundlagen der Mathematik, vol. 2. Springer (1939)

Kleene, S.C.: Review of Functional Logic without Axioms or Primitive Rules of Inference. Journal of Symbolic Logic **13**(3), 173–174 (1948)

Kleene, S.C.: Review of On the Theory of Deduction, Part I and II, and The Trivialization of Mathematical Logic. Journal of Symbolic Logic **14**(1), 62–63 (1949)

McKinsey, J.C.C.: Review of Logic without Assumptions and New Foundations for Logic. Journal of Symbolic Logic **13**(2), 114–115 (1948)

Piecha, T., Schroeder-Heister, P. (eds.): Advances in Proof-Theoretic Semantics, *Trends in Logic*, vol. 43. Springer (2016). URL https://link.springer.com/content/pdf/10.1007%2F978-3-319-22686-6.pdf

Popper, K.R.: Logic Lecture Notes 1939/41. Unpublished typescript (1939/1941). Karl Popper-Sammlung, Universität Klagenfurt, Box 366, Folder 19.

Popper, K.R.: The Open Society and Its Enemies. Routledge, London (1945)

Popper, K.R.: Unpublished draft of a letter to Alexander Carr-Saunders (1946). Karl Popper-Sammlung, Universität Klagenfurt, Box 368, Folder 2.

Popper, K.R.: Functional Logic without Axioms or Primitive Rules of Inference. Koninklijke Nederlandse Akademie van Wetenschappen, Proceedings of the section of sciences **50**, 1214–1224 (1947a)

Popper, K.R.: Letter to R. Carnap, November 24th 1947 (1947b). Karl Popper-Sammlung, Universität Klagenfurt, Box 282, Folder 24.

Popper, K.R.: Logic without Assumptions. Proceedings of the Aristotelian Society **47**, 251–292 (1947c)

Popper, K.R.: New Foundations for Logic. Mind **56**(223), 193–235 (1947d)

Popper, K.R.: Unpublished letter to P. Bernays, March 3rd 1947 (1947e). ETH Zürich, Paul Bernays Collection, Hs 975 : 3650. Translation by the authors.

Popper, K.R.: Unpublished letter to P. Bernays, October 19th 1947 (1947f). ETH Zürich, Paul Bernays Collection, Hs 975 : 3651. Translation by the authors.

Popper, K.R.: Corrections and Additions to "New Foundations for Logic". Mind **57**(225), 69–70 (1948a)

Popper, K.R.: On the Theory of Deduction, Part I. Derivation and its Generalizations. Koninklijke Nederlandse Akademie van Wetenschappen, Proceedings of the section of sciences **51**, 173–183 (1948b)

Popper, K.R.: On the Theory of Deduction, Part II. The Definitions of Classical and of Intuitionist Negation. Koninklijke Nederlandse Akademie van Wetenschappen, Proceedings of the section of sciences **51**, 322–331 (1948c)

Popper, K.R.: Unpublished letter to P. Bernays, June 13th 1948 (1948d). ETH Zürich, Paul Bernays Collection, Hs 975 : 3654. Translation by the authors.

Popper, K.R.: The Trivialization of Mathematical Logic. In: Beth, E.W., Pos, H.J., Hollak, J.H.A. (eds.) Proceedings of the Tenth International Congress of Philosophy, vol. 1, pp. 722–727. North-Holland (1949)

Popper, K.R.: Part One: Autobiography of Karl Popper. In: Schilpp, P.A. (ed.) The Philosophy of Karl Popper, pp. 1–181. La Salle: Open Court (1974)

Popper, K.R.: Derivation and Demonstration in Propositional and Functional Logic (n.d.a). Karl Popper-Sammlung, Universität Klagenfurt, Box 36, Folder 20.

Popper, K.R.: Table of contents for a textbook in logic (n.d.b). Karl Popper-Sammlung, Universität Klagenfurt, Box 371, Folder 1.

Popper, K.R.: The Propositional and Functional Logic of Derivation and of Demonstration (n.d.c). Karl Popper-Sammlung, Universität Klagenfurt, Box 36, Folder 21.

Popper, K.R.: Unnamed typescript on the distinction between derivation and demonstration (n.d.d). Karl Popper-Sammlung, Universität Klagenfurt, Box 36, Folder 20.

Popper, K.R., Bernays, P.: On Systems of Rules of Inference (n.d.). Karl Popper-Sammlung, Universität Klagenfurt, Box 36, Folder 13.

Quine, W.V.O.: Mathematical Logic. W. W. Norton & Company, New York (1940)

Schroeder-Heister, P.: Popper's theory of deductive inference and the concept of a logical constant. History and Philosophy of Logic **5**, 79–110 (1984)

Schroeder-Heister, P.: Popper's structuralist theory of logic. In: Miller, D., Jarvie, I., Milford, K. (eds.) Karl Popper: A Centenary Assessment. Vol III: Science, pp. 17–36. Ashgate, Aldershot (2006)

Schroeder-Heister, P.: Proof-Theoretic Semantics. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy (2018). URL https://plato.stanford.edu/entries/proof-theoretic-semantics/

Seldin, J.P.: Curry's attitude towards Popper. In: Proceedings of the Annual Meeting of the Canadian Society for History and Philosophy of Mathematics, vol. 21, University of British Columbia, Vancouver, BC, 1–3 June 2008, pp. 142–149 (2008)

Whitehead, A.N., Russell, B.: Principia Mathematica. 2nd edn. Cambridge University Press (1925/1927)