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Inversion and the Admissibility of Logical Rules

Wagner de Campos Sanz[1] and Thomas Piecha[2]*

[1] Universidade Federal de Goiás, Faculdade de Filosofia, Campus II, Goiânia, GO, Brazil, CEP 74001-970 – sanz@fchf.ufg.br
[2] Universität Tübingen, Wilhelm-Schickard-Institut für Informatik, Sand 13, 72076 Tübingen, Germany – piecha@informatik.uni-tuebingen.de

Abstract. The inversion principle expresses a relationship between left and right introduction rules for logical constants. Hallnäs and Schroeder-Heister [2] presented the principle of definitional reflection as a means of capturing the idea embodied in the inversion principle. Using the principle of definitional reflection, we show for minimal propositional logic that the left introduction rules are admissible when the right introduction rules are given as the definition of logical constants, and vice versa.

Keywords: Proof theory, inversion principle, admissibility, logical rules.

1 Inversion Principle and Definitional Reflection

The idea underlying the inversion principle for logical rules can be found in certain remarks made by Gentzen [1, p. 189] to the effect that the logical constants are defined by the right introduction rules whereas the left introduction rules are then only consequences thereof. The inversion principle itself was introduced by Lorenzen [3] without being restricted to logical rules, and it was later formulated by Prawitz [4] for logical rules in the context of natural deduction. Following Schroeder-Heister [5], the inversion principle is based on the idea that if we have certain defining rules $A \Leftarrow B_1^1, \ldots, B_{n_1}^1, \ldots, A \Leftarrow B_1^k, \ldots, B_{n_k}^k$ for some atom A, then a rule $C \Leftarrow A$ with premiss A and conclusion C is justified if C is consequence of each set of premisses Γ_i of rules defining A, where $\Gamma_i = B_1^i, \ldots, B_{n_i}^i$, i.e., if C is derivable from each Γ_i, then C is derivable from A.

This principle can be stated by means of a sequent calculus inference (cf. Hallnäs and Schroeder-Heister [2]) and is then called the *principle of definitional reflection* ($\mathcal{D}\vdash$):

$$(\mathcal{D}\vdash)\ \frac{\Delta_1, \Gamma_1 \vdash C \quad \ldots \quad \Delta_k, \Gamma_k \vdash C}{\Delta_1, \ldots, \Delta_k, A \vdash C} \qquad \text{(definitional reflection)}$$

where for $\mathcal{D}(A) = \{\Gamma_1, \ldots, \Gamma_k\}$ being the sets of premisses in rules defining A the following proviso has to be observed: For any substitution σ of variables by terms, the application of definitional reflection is restricted to the cases where $\mathcal{D}(A^\sigma) \subseteq (\mathcal{D}(A))^\sigma$.

Definitional reflection has the form of a left introduction rule for atoms A defined by rules with sets of premisses $\Gamma_1, \ldots, \Gamma_k$, and is thus a way of stating the inversion principle for definitions. This principle is accompanied by a

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right introduction rule for atoms A defined by a rule with premisses $B_1^i, \ldots, B_{n_i}^i$ (*definitional closure*) which, however, is not needed in what follows. In addition to definitional reflection we use the structural inferences *identity* (Id), *thinning* (Thin) and *cut* (Cut) in our logical framework:

$$(\mathrm{Id})\ \frac{}{A \vdash A} \qquad (\mathrm{Thin})\ \frac{\Delta \vdash A}{B, \Delta \vdash A} \qquad (\mathrm{Cut})\ \frac{\Delta \vdash C \quad C, \Sigma \vdash A}{\Delta, \Sigma \vdash A}$$

2 Admissibility of Logical Rules

Concerning a given rule R and a given definition $\mathcal{D}$, rule R is *admissible* in $\mathcal{D}$, if the relation of being producible for $\mathcal{D}$ is not enlarged by adding R to $\mathcal{D}$, yielding the extended system $\mathcal{D}+R$. Let $\Vdash_{\mathcal{D}+R} A$ denote the producibility of A in definition $\mathcal{D}$ with added rule R. Then R is admissible in $\mathcal{D}$ if for every A the implication "if $\Vdash_{\mathcal{D}+R} A$, then $\Vdash_{\mathcal{D}} A$" holds. The principle of definitional reflection can be interpreted as a principle for admissibility if sequents $B_1, \ldots, B_n \vdash A$ are interpreted as stating the admissibility of rules $A \Leftarrow B_1, \ldots, B_n$ relative to a given definition $\mathcal{D}$. Consider the rule $C \Leftarrow A, \Delta_1, \ldots, \Delta_k$ which corresponds to the conclusion of definitional reflection. Then A was derived by a rule $A \Leftarrow B_1^i, \ldots, B_{n_i}^i$, for some i, in the last step and $B_1^i, \ldots, B_{n_i}^i$ were derived in previous steps (likewise for Δ_i). Thus, if the rules $C \Leftarrow B_1^i, \ldots, B_{n_i}^i, \Delta_i$ (corresponding to the premisses of definitional reflection) are admissible, then the rule corresponding to the conclusion of definitional reflection is admissible as well since all consequences C following from $B_1^i, \ldots, B_{n_i}^i, \Delta_i$ should be consequences of A.

Sequent calculus rules can be understood as definitions for logical constants. For the right and left introduction rules we use the following representation for object language sequents $\mathfrak{s}$, called *o-sequents*: $\begin{smallmatrix}\Omega\\ \bigtriangledown\\ A\end{smallmatrix}$ ("A follows from Ω"). The o-sequents are to be distinguished from the sequents in the framework, which are called *f-sequents* and are expressed with the turnstile '$\vdash$'. Finite sets of o-sequents are denoted by $\mathfrak{S}$. Our aim is to represent sequent style minimal propositional logic. This is why o-sequents have exactly one formula at the bottom; it corresponds to the succedent of sequents. What is written on top (corresponding to the antecedent of sequents) is either a (possibly empty) finite multiset of formulas or a comma-separated list of such sets, the comma representing multiset union. The sequent symbol '$\bigtriangledown$' represents the relation of deductive consequence. Hence, the logical constants will be defined in the context of deductive consequence.

The properties of the usual deductive consequence relation are captured in sequent calculus by the inferences *identity*, *thinning* and *cut*. The following inferences (o-Id), (o-Thin) and (o-Cut) express these properties for o-sequents, and are added to the framework:

$$\frac{}{\vdash \begin{matrix}A\\ \bigtriangledown\\ A\end{matrix}}\ (\text{o-Id}) \qquad \frac{\mathfrak{S} \vdash \begin{matrix}\Omega\\ \bigtriangledown\\ A\end{matrix}}{\mathfrak{S} \vdash \begin{matrix}\Omega, B\\ \bigtriangledown\\ A\end{matrix}}\ (\text{o-Thin}) \qquad \frac{\mathfrak{S}_1 \vdash \begin{matrix}\Omega\\ \bigtriangledown\\ C\end{matrix} \quad \mathfrak{S}_2 \vdash \begin{matrix}C, \Psi\\ \bigtriangledown\\ A\end{matrix}}{\mathfrak{S}_1, \mathfrak{S}_2 \vdash \begin{matrix}\Omega, \Psi\\ \bigtriangledown\\ A\end{matrix}}\ (\text{o-Cut})$$

Thus the framework comprises definitional reflection ($\mathcal{D}\vdash$), the structural inferences for f-sequents (Id), (Thin) and (Cut), and the structural inferences for o-sequents (o-Id), (o-Thin) and (o-Cut).

2.1 Admissibility of Left Introduction Rules

For the right introduction rules, i.e., rules for the introduction of a logical constant in the bottom of an o-sequent, we can derive f-sequents of the form $\mathfrak{S}\vdash\mathfrak{s}$ representing the left introduction rules, i.e., rules for the introduction of a logical constant in the top of an o-sequent, inside the framework by using the corresponding definitional reflections as depicted in the following table.

RIGHT INTRODUCTION RULE	DEFINITIONAL REFLECTION
$\mathcal{D}_\wedge\left\{\begin{array}{c}\Omega\\ \triangledown\\ A\wedge B\end{array}\Leftarrow\begin{array}{c}\Omega\\ \triangledown\\ A\end{array},\begin{array}{c}\Omega\\ \triangledown\\ B\end{array}\right.$	$(\mathcal{D}_\wedge\vdash)\,\dfrac{\mathfrak{S},\begin{array}{c}\Omega\\ \triangledown\\ A\end{array},\begin{array}{c}\Omega\\ \triangledown\\ B\end{array}\vdash\mathfrak{s}}{\mathfrak{S},\begin{array}{c}\Omega\\ \triangledown\\ A\wedge B\end{array}\vdash\mathfrak{s}}$
$\mathcal{D}_\vee\left\{\begin{array}{l}\begin{array}{c}\Omega\\ \triangledown\\ A\vee B\end{array}\Leftarrow\begin{array}{c}\Omega\\ \triangledown\\ A\end{array}\\ \begin{array}{c}\Omega\\ \triangledown\\ A\vee B\end{array}\Leftarrow\begin{array}{c}\Omega\\ \triangledown\\ B\end{array}\end{array}\right.$	$(\mathcal{D}_\vee\vdash)\,\dfrac{\mathfrak{S}_1,\begin{array}{c}\Omega\\ \triangledown\\ A\end{array}\vdash\mathfrak{s}\qquad\mathfrak{S}_2,\begin{array}{c}\Omega\\ \triangledown\\ B\end{array}\vdash\mathfrak{s}}{\mathfrak{S}_1,\mathfrak{S}_2,\begin{array}{c}\Omega\\ \triangledown\\ A\vee B\end{array}\vdash\mathfrak{s}}$
$\mathcal{D}_\rightarrow\left\{\begin{array}{c}\Omega\\ \triangledown\\ A\rightarrow B\end{array}\Leftarrow\begin{array}{c}\Omega,A\\ \triangledown\\ B\end{array}\right.$	$(\mathcal{D}_\rightarrow\vdash)\,\dfrac{\mathfrak{S},\begin{array}{c}\Omega,A\\ \triangledown\\ B\end{array}\vdash\mathfrak{s}}{\mathfrak{S},\begin{array}{c}\Omega\\ \triangledown\\ A\rightarrow B\end{array}\vdash\mathfrak{s}}$

As an example, we show the admissibility of the left conjunction introduction rule by deriving its corresponding f-sequent (for the conjunct A; likewise for B):

$$\dfrac{\dfrac{}{\vdash\begin{array}{c}A\wedge B\\ \triangledown\\ A\wedge B\end{array}}(\text{o-Id})\qquad (\mathcal{D}_\wedge\vdash)\dfrac{(\text{Thin})\dfrac{\dfrac{(\text{Id})\dfrac{}{\begin{array}{c}A\wedge B\\ \triangledown\\ A\end{array}\vdash\begin{array}{c}A\wedge B\\ \triangledown\\ A\end{array}}\qquad(\text{Id})\dfrac{}{\begin{array}{c}\Theta,A\\ \triangledown\\ C\end{array}\vdash\begin{array}{c}\Theta,A\\ \triangledown\\ C\end{array}}}{\begin{array}{c}A\wedge B\\ \triangledown\\ A\end{array},\begin{array}{c}\Theta,A\\ \triangledown\\ C\end{array}\vdash\begin{array}{c}\Theta,A\wedge B\\ \triangledown\\ C\end{array}}(\text{o-Cut})}{\begin{array}{c}\Theta,A\\ \triangledown\\ C\end{array},\begin{array}{c}A\wedge B\\ \triangledown\\ A\end{array},\begin{array}{c}A\wedge B\\ \triangledown\\ B\end{array}\vdash\begin{array}{c}\Theta,A\wedge B\\ \triangledown\\ C\end{array}}}{\begin{array}{c}\Theta,A\\ \triangledown\\ C\end{array},\begin{array}{c}A\wedge B\\ \triangledown\\ A\wedge B\end{array}\vdash\begin{array}{c}\Theta,A\wedge B\\ \triangledown\\ C\end{array}}}{\begin{array}{c}\Theta,A\\ \triangledown\\ C\end{array}\vdash\begin{array}{c}\Theta,A\wedge B\\ \triangledown\\ C\end{array}}(\text{Cut})$$

Similar derivations can be given for disjunction and for implication.

2.2 Admissibility of Right Introduction Rules

For each left introduction rule given by $\mathcal{D}^{\wedge}$, $\mathcal{D}^{\vee}$ and $\mathcal{D}^{\rightarrow}$ the admissibility of the corresponding right introduction rule can be shown by using the respective definitional reflections $(\mathcal{D}^{\wedge}\vdash)$, $(\mathcal{D}^{\vee}\vdash)$ and $(\mathcal{D}^{\rightarrow}\vdash)$ as shown below:

LEFT INTRODUCTION RULE	DEFINITIONAL REFLECTION
$\mathcal{D}^{\wedge}\left\{\begin{array}{l} \begin{array}{c}\Omega, A\wedge B\\ \bigtriangledown\\ C\end{array} \Leftarrow \begin{array}{c}\Omega, A\\ \bigtriangledown\\ C\end{array} \\ \begin{array}{c}\Omega, A\wedge B\\ \bigtriangledown\\ C\end{array} \Leftarrow \begin{array}{c}\Omega, B\\ \bigtriangledown\\ C\end{array}\end{array}\right.$	$(\mathcal{D}^{\wedge}\vdash)\,\dfrac{\mathfrak{S}_1, \begin{array}{c}\Omega,A\\ \bigtriangledown\\ C\end{array} \vdash \mathfrak{s} \qquad \mathfrak{S}_2, \begin{array}{c}\Omega,B\\ \bigtriangledown\\ C\end{array}\vdash\mathfrak{s}}{\mathfrak{S}_1,\mathfrak{S}_2, \begin{array}{c}\Omega, A\wedge B\\ \bigtriangledown\\ C\end{array}\vdash\mathfrak{s}}$
$\mathcal{D}^{\vee}\left\{ \begin{array}{c}\Omega, A\vee B\\ \bigtriangledown\\ C\end{array} \Leftarrow \begin{array}{c}\Omega, A\\ \bigtriangledown\\ C\end{array} , \begin{array}{c}\Omega, B\\ \bigtriangledown\\ C\end{array}\right.$	$(\mathcal{D}^{\vee}\vdash)\,\dfrac{\mathfrak{S}, \begin{array}{c}\Omega,A\\ \bigtriangledown\\ C\end{array} , \begin{array}{c}\Omega,B\\ \bigtriangledown\\ C\end{array}\vdash\mathfrak{s}}{\mathfrak{S}, \begin{array}{c}\Omega, A\vee B\\ \bigtriangledown\\ C\end{array}\vdash\mathfrak{s}}$
$\mathcal{D}^{\rightarrow}\left\{ \begin{array}{c}\Omega, A\rightarrow B\\ \bigtriangledown\\ C\end{array} \Leftarrow \begin{array}{c}\Omega\\ \bigtriangledown\\ A\end{array} , \begin{array}{c}\Omega, B\\ \bigtriangledown\\ C\end{array}\right.$	$(\mathcal{D}^{\rightarrow}\vdash)\,\dfrac{\mathfrak{S}, \begin{array}{c}\Omega\\ \bigtriangledown\\ A\end{array} , \begin{array}{c}\Omega,B\\ \bigtriangledown\\ C\end{array}\vdash\mathfrak{s}}{\mathfrak{S}, \begin{array}{c}\Omega, A\rightarrow B\\ \bigtriangledown\\ C\end{array}\vdash\mathfrak{s}}$

As an example, we show the admissibility of the right implication introduction rule by deriving its corresponding f-sequent:

$$\dfrac{\dfrac{}{\vdash \begin{array}{c}A\rightarrow B\\ \bigtriangledown\\ A\rightarrow B\end{array}}(\text{o-Id}) \qquad (\mathcal{D}^{\rightarrow}\vdash)\dfrac{\dfrac{(\text{Id})\dfrac{}{\begin{array}{c}\\ \bigtriangledown\\ A\end{array}\vdash\begin{array}{c}\\ \bigtriangledown\\ A\end{array}} \qquad \dfrac{(\text{Id})\dfrac{}{\begin{array}{c}\Theta,A\\ \bigtriangledown\\ B\end{array}\vdash\begin{array}{c}\Theta,A\\ \bigtriangledown\\ B\end{array}} \qquad (\text{Id})\dfrac{}{\begin{array}{c}B\\ \bigtriangledown\\ A\rightarrow B\end{array}\vdash\begin{array}{c}B\\ \bigtriangledown\\ A\rightarrow B\end{array}}}{\begin{array}{c}\Theta,A\\ \bigtriangledown\\ B\end{array} , \begin{array}{c}B\\ \bigtriangledown\\ A\rightarrow B\end{array}\vdash\begin{array}{c}\Theta,A\\ \bigtriangledown\\ A\rightarrow B\end{array}}(\text{o-Cut})}{\begin{array}{c}\Theta,A\\ \bigtriangledown\\ B\end{array} , \begin{array}{c}\\ \bigtriangledown\\ A\end{array} , \begin{array}{c}B\\ \bigtriangledown\\ A\rightarrow B\end{array}\vdash\begin{array}{c}\Theta\\ \bigtriangledown\\ A\rightarrow B\end{array}}(\text{o-Cut})}{\begin{array}{c}\Theta,A\\ \bigtriangledown\\ B\end{array} , \begin{array}{c}A\rightarrow B\\ \bigtriangledown\\ A\rightarrow B\end{array}\vdash\begin{array}{c}\Theta\\ \bigtriangledown\\ A\rightarrow B\end{array}}}{\begin{array}{c}\Theta,A\\ \bigtriangledown\\ B\end{array}\vdash\begin{array}{c}\Theta\\ \bigtriangledown\\ A\rightarrow B\end{array}}(\text{Cut})$$

Similar derivations can be given for conjunction and for disjunction.

Therefore not only the left introduction rules can be shown to be admissible for given right introduction rules, but the right introduction rules can be shown to be admissible for given left introduction rules as well.

2.3 Inversion versus Eliminability of (Cut)

Besides definitional reflection, the structural inferences for f-sequents (Id), (Thin) and (Cut) as well as the structural inferences for o-sequents (o-Id), (o-Thin) and (o-Cut) have to be used in the derivations showing admissibility of the left resp. right introduction rules for given right resp. left introduction rules (the inference (o-Thin) was not used in the two derivations shown above, but has to be used in the derivations of the f-sequents corresponding to the left disjunction and right conjunction introduction rules). We observe that (Cut) cannot be eliminated from those derivations, and is therefore not eliminable in general if only either left or right introduction rules are given for minimal propositional logic. If not only definitions for either left or right introduction rules are considered, but if the whole system of minimal propositional logic comprising left and right introduction rules is given (i.e. $\mathcal{D}_\wedge$, $\mathcal{D}_\vee$ and $\mathcal{D}_\rightarrow$ together with $\mathcal{D}^\wedge$, $\mathcal{D}^\vee$ and $\mathcal{D}^\rightarrow$), then the inversions of the respective rules are already given in the definitions, and inversion by definitional reflection can be dispensed with. In this case the eliminability of (Cut) is immediate.

3 Conclusion

Given the admissibility results shown above, it seems questionable that the right introduction rules have any kind of privilege over the left introduction rules concerning the definition of logical constants, or vice versa. The logical constants of minimal propositional logic can be defined by right introduction rules as well as by left introduction rules. If the right introduction rules are given as definitions, then the left introduction rules are consequences of them in the sense of being admissible relative to the given definitions, and if the left introduction rules are given as definitions, then the right introduction rules are consequences of them in the same sense of being admissible.

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