# Advanced Mathematical Methods

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## 5 Mathematical Statistics

Prof. Dr. Thomas Dimpfl

*Department of Statistics, Econometrics and Empirical Economics*

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Wirtschafts- und Sozialwissenschaftliche Fakultät

# Outline: Mathematical Statistics

# Readings

- A. Papoulis and A. U. Pillai. *Probability, Random Variables and Stochastic Processes.*
  Mc Graw Hill, fourth edition, 2002, Chapter 6

# Online References

– none –

# 5.13 The bivariate normal distribution

## Definition: Bivariate normal distribution

Two random variables $X_1$ and $X_2$ are jointly normally distributed if they are described by the joint pdf

$$f_{\underline{X}}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot \exp\left[-\frac{1}{2}q(x_1, x_2)\right]$$

where

$$q(x_1, x_2) = \frac{1}{1-\rho^2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

# 5.13 The bivariate normal distribution

if $(X_1, X_2) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. then

- $f_{X_1}(x_1) \sim N(\mu_1, \sigma_1^2)$
  $f_{X_2}(x_2) \sim N(\mu_2, \sigma_2^2)$
- $f_{X_1|X_2} \sim N(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2), \sigma_1^2(1 - \rho^2))$

  $f_{X_2|X_1} \sim N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1), \sigma_2^2(1 - \rho^2))$

# 5.14 Multivariate Distributions

$\boldsymbol{x}$ a random vector with joint density $f(\boldsymbol{x})$

$$F(\boldsymbol{x}) = \int\limits_{-\infty}^{x_n} \int\limits_{-\infty}^{x_{n-1}} \cdots \int\limits_{-\infty}^{x_1} f(\boldsymbol{t}) dt_1 dt_2 \ldots dt_{n-1} dt_n$$

Expected Value:

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = \begin{pmatrix} E[X_1] \\ \vdots \\ E[X_n] \end{pmatrix}$$

# 5.14 Multivariate Distributions

## Covariance Matrix

$$E\left[(\boldsymbol{x}-\boldsymbol{\mu})(\boldsymbol{x}-\boldsymbol{\mu})'\right]$$

$$= \begin{pmatrix} (x_1-\mu_1)(x_1-\mu_1) & (x_1-\mu_1)(x_2-\mu_2) & \dots & (x_1-\mu_1)(x_n-\mu_n) \\ (x_2-\mu_2)(x_1-\mu_1) & (x_2-\mu_2)(x_2-\mu_2) & \dots & (x_2-\mu_2)(x_n-\mu_n) \\ \vdots & & & \\ (x_n-\mu_n)(x_1-\mu_1) & (x_n-\mu_2)(x_n-\mu_2) & \dots & (x_n-\mu_n)(x_n-\mu_n) \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \dots \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \dots \sigma_{2n} \\ \vdots & & \\ \sigma_{n1} & \dots & \dots \sigma_n^2 \end{pmatrix} = E\left[\boldsymbol{x}\boldsymbol{x}'\right] - \boldsymbol{\mu}\boldsymbol{\mu}' = \boldsymbol{\Sigma}$$

# 5.14 Multivariate Distributions

Linear Transformation: sum of $n$ random variables $\sum_{i=1}^{n} a_i x_i$

$$
\begin{aligned}
E[a_1 x_1 + a_2 x_2 + \ldots a_n x_n] &= E[\mathbf{a}'\mathbf{x}] \\
&= \mathbf{a}' E[\mathbf{x}] = \mathbf{a}'\boldsymbol{\mu} \\
Var[\mathbf{a}'\mathbf{x}] &= E[(\mathbf{a}'\mathbf{x} - E[\mathbf{a}'\mathbf{x}])^2] \\
&= E[(\mathbf{a}'(\mathbf{x} - E[\mathbf{x}])^2] \\
&= E[(\mathbf{a}'(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})'\mathbf{a}] \\
&= \mathbf{a}' E[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})']\mathbf{a} \\
&= \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij}
\end{aligned}
$$

# 5.14 Multivariate Distributions

## Linear transformation: $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$

$i$-th element in $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ is $y_i = \boldsymbol{a_i}\boldsymbol{x}$ with $\boldsymbol{a_i}$ $i$-th row in $\boldsymbol{A}$

$\Rightarrow E[y_i] = E[\boldsymbol{a_i}\boldsymbol{x}] = \boldsymbol{a_i}\boldsymbol{\mu}$ as before

$$
\begin{aligned}
E[\boldsymbol{y}] &= E[\boldsymbol{A}\boldsymbol{x}] = \boldsymbol{A}E[\boldsymbol{x}] = \boldsymbol{A}\boldsymbol{\mu} \\
Var[\boldsymbol{y}] &= E[(\boldsymbol{y} - E[\boldsymbol{y}])(\boldsymbol{y} - E[\boldsymbol{y}])'] \\
&= E[(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{\mu})(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{\mu})'] \\
&= E[(\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{\mu})[(\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{\mu})]'] \\
&= E[\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{\mu})(\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{A}'] \\
&= \boldsymbol{A}E[(\boldsymbol{x} - \boldsymbol{\mu})(\boldsymbol{x} - \boldsymbol{\mu})']\boldsymbol{A}' = \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}'
\end{aligned}
$$

# 5.15 Modes of Stochastic Convergence

- Convergence in probability: $\underset{p}{\rightarrow}$
- Convergence almost surely: $\underset{a.s.}{\rightarrow}$
- Convergence in mean square: $\underset{m.s.}{\rightarrow}$
- Convergence in distribution: $\underset{d}{\rightarrow}$

$\{z_n\}$: sequence of random variables
$\{\mathbf{z}_n\}$: sequence of random vectors

# 5.15 Modes of Stochastic Convergence

## Convergence in probability

A sequence $\{z_n\}$ converges in probability to a constant $\alpha$ if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(|z_n - \alpha| > \varepsilon\right) = 0$$

short notation: $\text{plim } z_n = \alpha$ or $z_n \underset{p}{\longrightarrow} \alpha$ or $z_n - \alpha \underset{p}{\longrightarrow} 0$

Extends to random vectors:

$$\text{If} \qquad \lim_{n\to\infty} \mathbb{P}\left(|z_{kn} - \alpha_k| > \varepsilon\right) = 0 \quad \forall \quad k = 1, 2, ..., K,$$

then $\mathbf{z}_n \underset{p}{\longrightarrow} \boldsymbol{\alpha}$ (element-wise convergence).

# 5.15 Modes of Stochastic Convergence

## Convergence almost surely

A sequence $\{z_n\}$ converges almost surely to a constant $\alpha$ if

$$\mathbb{P}\left(\lim_{n\to\infty} z_n = \alpha\right) = 1$$

short notation: $z_n \xrightarrow[a.s.]{} \alpha$.

<u>Extends to random vectors</u>:

$$\text{If} \qquad \mathbb{P}\left(\lim_{n\to\infty} z_{kn} = \alpha_k\right) = 1 = 0 \quad \forall \quad k = 1, 2, ..., K,$$

then $\mathbf{z}_n \xrightarrow[a.s.]{} \boldsymbol{\alpha}$ (element-wise convergence).

# 5.15 Modes of Stochastic Convergence

## Convergence in mean square

A sequence $\{z_n\}$ converges in mean square to a constant $\alpha$ if

$$\lim_{n \to \infty} \mathbb{E}\left[(z_n - \alpha)^2\right] = 0$$

short notation: $z_n \xrightarrow[m.s.]{} \alpha$

Convergence in mean square implies convergence in probability.

# 5.15 Modes of Stochastic Convergence

## Convergence in distribution

A sequence $\{z_n\}$ converges in distribution to a random variable $z$ if

$$z_n \xrightarrow[d]{} z$$

i.e., if the c.d.f. of $z_n$ converges to the c.d.f. of $z$ at each point of continuity.